

**Comprehensive Exam – Linear Algebra
Fall 2004**

1. (a) Determine if each of the following statements is TRUE or FALSE by giving a short proof or a counterexample. Assume below that the sets are finite and vector spaces are finite dimensional.

- (i) If S is a linearly dependent set of vectors then each element of S is a linear combination of the other elements of S .
 - (ii) The intersection of any two subspaces of a vector space V is also a subspace of V .
 - (iii) Let S be a subset of a vector space V . If S spans V , then each vector in V can be written as a linear combination of elements of S in only one way.
- (b) Construct two bases for R^4 such that the only vectors common to both sets are $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$. You MUST show that each set form a basis for R^4 .

2. (a) Suppose $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for a vector space V .

- (i) Show that $\gamma = \{v_1 + v_2, v_2 + v_3, \dots, v_{n-1} + v_n, v_n\}$ is also a basis for V .
 - (ii) Prove that the linear transformation $T: V \rightarrow V$ such that $T(\beta) = \gamma$ is an isomorphism of V .
- (b) Let $T: R^2 \rightarrow R^2$ be a linear transformation defined by $T(x, y) = (-5x + 9y, -4x + 7y)$.
- (i) Find the matrix representation $B = [T]_\beta$ relative to the ordered basis $\beta = \{(3, 2), (1, 1)\}$.
 - (ii) Show that for any positive integer n , $B^n - I = n(B - I)$ where I is the identity matrix.
 - (iii) Suppose $A = [T]_\alpha$ where $\alpha = \{(1, 0), (0, 1)\}$ is the standard basis for R^2 . Evaluate A^n and verify that $\det(A^n) = \det(A)^n$.

3. (a) Suppose an inner product is defined on the complex vector space C^n by $\langle x, y \rangle = y^* H x$, $x, y \in C^n$ where H is a complex, $n \times n$ matrix and y^* is the matrix adjoint of y . Prove that H is a Hermitian matrix with positive diagonal entries.

(b) A norm on the vector space $M_{n \times n}(C)$ of complex, $n \times n$ matrices is defined by the real valued function $\|A\| = \max_{1 \leq i, j \leq n} n|A_{ij}|$, $A \in M_{n \times n}(C)$. Prove that the given norm satisfies the following.

- (i) $\|A\| \geq 0$ and $\|A\| = 0$ if and only if $A = 0$.
- (ii) $\|cA\| = |c|\|A\|$ for complex scalars c .
- (iii) $\|A + B\| \leq \|A\| + \|B\|$, $A, B \in M_{n \times n}(C)$.
- (iv) $\|AB\| \leq \|A\|\|B\|$, $A, B \in M_{n \times n}(C)$.
- (v) If A is invertible then $\|A^{-1}\| \geq \frac{n}{\|A\|}$.

Please turn over

4. Given that $p(t) = a_0 + a_1t + \dots + a_{n-1}t^{n-1} + a_nt^n$ is the characteristic polynomial of an invertible, $n \times n$ matrix A .

(a) Show that $g(t) = \frac{(-1)^n}{\det(A)}(a_n + a_{n-1}t + \dots + a_1t^{n-1} + a_0t^n)$ is the characteristic polynomial of A^{-1} .

(b) Prove that λ is an eigenvalue of A with multiplicity m if and only if λ^{-1} is an eigenvalue of A^{-1} with the same multiplicity.

(c) If A is a unitary matrix and λ is an eigenvalue of A then show that $|\lambda| = 1$.

5. Consider the vector space V spanned by the basis set $\beta = \{e^x, xe^x, x^2e^x, e^{2x}\}$ of real valued functions. Let T be a linear operator on V defined by $T(f) = \frac{df}{dx}$, $f \in V$.

(a) Find a Jordan canonical form J and a Jordan canonical basis γ for T .

(b) Find the minimal polynomial of T . Hence show that V is the solution space of a fourth order, linear differential equation. Determine this differential equation explicitly.