Analysis Exam Aug 2003

1. (a) $\sum \frac{1}{e^k}$ ratio test
or root test
or integral test

$\sum \frac{1}{3k+1}$ comparison with harmonic series

(b) Weierstrass ~~M~~-test
or unif Cauchy convergence criterion

2. $\lim f_n(x) = \begin{cases} 1 & x=0 \\ 0 & x>0. \end{cases}$

(a)

(b) no limit is not continuous
~~If~~ convergence were uniform then limit would be continuous
by continuity of $f_n(x)$

3. (a) If a_n ~~is~~ monotone and bounded then a_n converges

(b) a_n is monotone increasing
$a_n \le \sum_{k=1}^{\infty} \frac{1}{3^k} \le \frac{1}{1-\frac{1}{3}} = 3/2.$

Therefore a_n is bounded above. QED

4. x_n is Cauchy if $\forall \varepsilon > 0 \; \exists N_\varepsilon$ s.t. $n, m \geq N_\varepsilon \Rightarrow$

$$d(x_n, x_m) < \varepsilon.$$

K is compact if every sequence has a further subsequence that converges in K.

Let x_n be a Cauchy sequence in a compact set K.

Then since K is compact we have $\exists$ ~~N st. then st~~ subsequence x_{n_k} and x ~~and K st~~ s.t. $\forall \varepsilon > 0 \; \exists K_\varepsilon$ with

$$d(x_{n_k}, x) < \varepsilon \text{ for all } k \geq K_\varepsilon$$

Let now $M = \max\{K_{\varepsilon/2}, N_{\varepsilon/2}\}$ Then for $k \geq M$

$$d(x_k, x) \leq d(x_k, x_{n_k}) + d(x_{n_k}, x)$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Q.E.D.

5.(a) f is a contraction if $\exists\ a\ \lambda < 1$ st. $d(f(x), f(y)) \leq \lambda\, d(x,y)$.

(b) $f(x) - f(y) = f'(u)\,(x-y)$

so $|f(x) - f(y)| \leq \lambda\,|x-y|$. QED.

6. (a) $|f_n(x) - f(x)| < \varepsilon$ for all $x \in [0,1]$ and $n \geq N_\varepsilon$.

Therefore since by uniform convergence f is ~~integrable~~ continuous and therefore integrable we have

$f_n(x) - f(x)$ is integrable and by $-\varepsilon < f_n(x) - f(x) < \varepsilon$

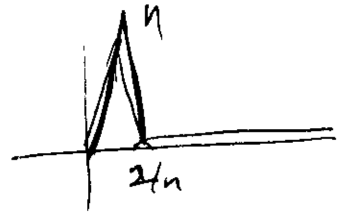
we have

$$-\varepsilon < \int f_n(x) - f(x)\,dx < \varepsilon$$

for all $n \geq N_\varepsilon$.

(a) Therefore $\left| \int f_n(x)\,dx - \int f(x)\,dx \right| < \varepsilon$ for $n \geq N_\varepsilon$.

QED.

(b) Let $f_n(x) =$ 

Then $f_n(x) \to f(x) \equiv 0$

but $\int f_n(x)dx = 1$ $\int f(x)dx = 0$

7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable.

Then by definition of local minimum $g(x) = f(x,0)$, ~~is minimized at x = 0~~. $: \mathbb{R} \to \mathbb{R}$ has a local minimum at $x = 0$

Therefore since $g'(0)$ exists with $g'(0) = \frac{\partial f}{\partial x}(0,0)$ we must have $g'(0) = 0$. Therefore $\frac{\partial f}{\partial x}(0,0) = 0$.

Similar argument for $\frac{\partial f}{\partial y}(0,0)$.

Let $f(x,y) = x^2 - y^2$

Then $\frac{\partial f}{\partial x}(0,0) = 0 \quad \frac{\partial f}{\partial y}(0,0) = 0$

But f has no local max or min at $(0,0)$

Indeed $g(x) = f(x,0) = x^2 > 0$ for $x \neq 0$

$h(y) := f(0,y) = -y^2 < 0$ for $y \neq 0$

So in any nbd of $(0,0)$ there are both positive and negative values of f.