

Department of Mathematics
Analysis Comprehensive Exam - Fall 2003

1. a) Determine whether the following series converge or diverge. Justify your answer.

$$\sum_{k=1}^{\infty} \frac{1}{e^k}, \quad \sum_{k=1}^{\infty} \frac{1}{3k+1}.$$

- b) Prove the series

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

converges uniformly on $[-a, a]$ for any positive real constant a .

2. Let $f_n(x) = e^{-nx}$ for $n \geq 1$ and $x \geq 0$.
- a) Show that f_n has a pointwise limit on $[0, \infty)$.
- b) Does f_n converge uniformly? Support your claim with a rigorous argument.
3. a) State the Monotone Convergence Theorem for sequences of real numbers $\{a_n\}_{n=1}^{\infty}$.
- b) Let $a_n = \sum_{k=1}^n \frac{1}{k} \frac{1}{3^k}$. Prove that $\lim_{n \rightarrow \infty} (a_n)$ exists.
4. Suppose that (x_n) is a Cauchy sequence in a compact metric space K . Show directly using the definitions of "Cauchy sequence" and "compact set" that the sequence converges in K .
5. a) Let X be a metric space with metric d and let $f : X \rightarrow X$. Define what it means for f to be a contraction on X .
- b) Use the Mean Value Theorem to show that if $f : R \rightarrow R$ has a derivative satisfying $|f'(x)| \leq \lambda$ for all x with $0 \leq \lambda < 1$, then f is a contraction on R .
6. Suppose that (f_n) is a sequence of continuous functions on $[0, 1]$ which converge to f on $[0, 1]$.
- a) If the convergence is uniform prove that

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = \int_0^1 f(x) dx.$$

- b) Give example in which convergence is not uniform and

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx \neq \int_0^1 f(x) dx.$$

7. a) Let f be a continuously differentiable function from R^2 to R . Show that if there is a local minimum at $(0, 0)$ then we must have $\frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = 0$.
- b) If $\frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = 0$, does this imply that there is a local extrema at $(0, 0)$? Prove it or give a counterexample.