

Fall 2003

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Comprehensive Exam - Linear Algebra

- 15 1. An  $n \times n$ , real matrix  $B$  is defined by :  $B_{ij} = 1, \forall i, j = 1 \dots n$ .
- 4 (a) Find a basis for the column space of  $B$ .
  - 4 (b) Find a basis for the null space of  $B$ .
  - 7 (c) Show that the eigenvalues of  $B$  are either 0 or  $n$ . Find an eigenvector corresponding to the eigenvalue  $n$ .

- 15 2. Let  $V$  and  $W$  be finite dimensional vector spaces and  $T : V \rightarrow W$  be a linear transformation.  $T$  is *one-to-one* iff  $T(x) = T(y) \Rightarrow x = y, \forall x, y \in V$ .  $T$  is *onto* iff the range  $R(T) = W$ .
- (a) Give an example (including a clear explanation) of a linear transformation which is:
- 4 (i) *one-to-one* but NOT onto
  - 4 (ii) *onto* but NOT *one-to-one*.
- 7 (b) Suppose  $\dim(V) = \dim(W)$ , then prove that  $T$  is *one-to-one* iff  $T$  is *onto*.

- 20 3. Consider the vector space  $M_{2 \times 2}(C)$  of all  $2 \times 2$ , complex matrices. Let  $A^*$  denote the adjoint of  $A \in M_{2 \times 2}(C)$  and  $Tr(A)$  denote its trace. Define  $W \subset M_{2 \times 2}(C)$  to be the set all  $2 \times 2$  matrices satisfying

$$W = \{X \in M_{2 \times 2}(C) : X^* = X, Tr(X) = 0\}.$$

- 6 (a) Prove that  $W$  is a subspace of  $M_{2 \times 2}(C)$ .
- 6 (b) Show that a basis for  $W$  is given by  $\beta = \{\sigma_1, \sigma_2, \sigma_3\}$  where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- 8 (c) Let  $[X]_\beta$  denote the coordinates of a vector  $X \in W$  with respect to the ordered basis  $\beta$  given above. Prove that  $\phi_\beta : W \rightarrow R^3$  defined by  $\phi_\beta(X) := [X]_\beta$  is an invertible linear transformation.

- 25 4. Let  $X, Y \in W$  as defined in Problem 3.

- 5 (a) Prove that  $\langle X, Y \rangle = Tr(XY)/2$  defines an inner product on  $W$ .

(b) Let  $\beta$  be the ordered basis for  $W$  given in Problem 3. Show that

- 4 (i)  $\beta$  is an *orthonormal* basis with respect to the above inner product.
- 4 (ii)  $\langle X, Y \rangle = \langle [X]_\beta, [Y]_\beta \rangle'$  where  $\langle, \rangle'$  is the standard inner product on  $R^3$ .

- 4 (c) Compute  $\langle X, X \rangle$  and deduce that every nonzero  $X \in W$  is an invertible matrix.

5. (a) Let  $m(\lambda)$  and  $p(\lambda)$  denote, respectively, the *minimal* and *characteristic* polynomials associated with an  $n \times n$  matrix  $A$ . Prove that the set of distinct zeros of  $m(\lambda)$  and  $p(\lambda)$  are the same.

(b) Suppose a  $5 \times 5$  matrix  $A$  has a minimal polynomial  $m(\lambda) = \lambda^2 + \lambda - 2$ .

(i) List the eigenvalues of  $A$  together with all possible algebraic multiplicities.

(ii) Show that  $A$  is invertible and express  $A^{-1}$  as a polynomial in  $A$ .

6. (a) Suppose the characteristic polynomial of a matrix  $A$  is given by  $p(\lambda) = \lambda^2(\lambda - 1)^2$ . List all possible inequivalent (not similar) Jordan canonical forms  $J$  such that  $A = PJP^{-1}$ .

(b) Given the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Determine the Jordan canonical form  $J$  and the matrix  $P$  such that  $A = PJP^{-1}$