

ANALYSIS COMPREHENSIVE - FALL 2000

Do each of the following problems.

- State the definition of a Cauchy sequence (s_n) in a metric space (X, d) .
 - Let $f_n(x) = \sum_{k=1}^n e^{-k} \cos^2(2\pi kx)$, $0 \leq x \leq 1$, be a sequence of functions (f_n) in the metric space C of continuous functions on $[0, 1]$ with the metric $d(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|$. Show that the sequence (f_n) is Cauchy.
 - Find all of the cluster points for the real sequence $(\cos(\frac{\pi k}{2}))_{k=1}^{\infty}$.
- Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function.
 - If (x_n) is a Cauchy sequence in \mathbb{R} , then show $(f(x_n))$ is a Cauchy sequence in \mathbb{R} .
 - If $K \subset \mathbb{R}$ is compact, then show $f(K)$ is compact.
 - Must $f^{-1}(K)$ be compact? (Prove or provide counter-example)
- Suppose the sequence of functions (f_n) with $f_n: [0, 1] \rightarrow \mathbb{R}$ continuous, converges uniformly to a function f .
 - Show that f is continuous.
 - Show that $\int_0^1 f_n(x) dx \rightarrow \int_0^1 f(x) dx$.
- State the Fundamental Theorem of Calculus.
 - Suppose that f is continuous on $[a, b]$. Let $F(x) = \int_a^x f(t) dt$ for $x \in [a, b]$. Show that if $F(x) = 0$ for all $x \in [a, b]$, then $f(x) = 0$ for all $x \in [a, b]$.
- Let $G: \mathbb{R}^n \rightarrow \mathbb{R}^m$.
 - Give the definition of the derivative $DG(x)$ for $x \in \mathbb{R}^n$.
 - Suppose that L is a linear map; that is $L(x+y) = L(x) + L(y)$ and $L(ax) = aL(x)$. Show that $DL(x) = L$.
- If a real valued function f is defined and continuous on the closed interval $[a, b]$, then show that f must be uniformly continuous.