Fall 2000

Comprehensive Examination – Linear Algebra

1. $B_1 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ and $B_2 = \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}$ are each bases of $\Re^2$.

   a) Determine the coordinates of the vectors in B_1 with respect to the basis B_2.
   b) Describe the matrix that changes B_2 coordinates to B_1 coordinates.

2. Suppose T is *any* non-singular linear transformation on $\Re^3$. I.e., T: $\Re^3 \to \Re^3$. Prove that there is a line in $\Re^3$ that is fixed by T. That is, there is some line ℓ such that if $\mathbf{x}$ is on ℓ, then $T(\mathbf{x})$ is also on ℓ.

3. Let $\mathcal{V}$ be the set of all polynomials in x with rational coefficients. It is given that $\mathcal{V}$ is a vector space over **Q**, the field of rational numbers. Let $\mathcal{W}$ be the subset of $\mathcal{V}$ consisting of all polynomials in x of degree less than or equal to 4 with rational coefficients.

   Let T, <u>restricted to $\mathcal{W}$</u>, be defined as follows: for any $p(x) \in \mathcal{W}$, $T[p(x)] = \dfrac{d^2}{dx^2} p(x)$.

   a) Show that $\mathcal{W}$ is a subspace of $\mathcal{V}$.
   b) What is the dimension of $\mathcal{W}$?
   c) Describe one basis for $\mathcal{W}$.
   d) Show that T is a linear transformation on $\mathcal{W}$.
   e) What is the kernel of T?
   f) Write down the matrix that represents T with respect to the basis found in c).

4. Suppose A is a matrix with characteristic polynomial:

   $$p(x) = x^3 + 4x^2 - 3x - 18 = (x+3)^2 (x-2).$$

   a) Find the characteristic polynomial of A^{-1}.
   b) Find the determinant and trace of A and A^{-1}.

5. In this problem, all matrices are square matrices over the field **C** of complex numbers.
   a) Define *Hermitian* matrix.
   b) For any A, show that A*A is Hermitian.
   c) Show that the diagonal of A*A consists of positive real numbers.
   d) Prove: If A is unitarily similar to B, then A*A is unitarily similar to B*B.

6. a) Consider the class of all 5 × 5 matrices that have the *characteristic* polynomial:

   $$p(x) = (x-2)^3 (x-1)^2.$$

   How many non-isomorphic Jordan Canonical Forms are similar to matrices in this class? Describe them.

   b) Consider the class of all 5 × 5 matrices that have the *mimimum* polynomial:

   $$p(x) = (x-2)^2 (x-1).$$

   How many non-isomorphic Jordan Canonical Forms are similar to matrices in this class? Describe them.

Do 2 of the following 3 problems on this page.

5. a) Define what it means for a subset E of the real numbers to have (Lebesgue) measure 0.

b) Show that the rational numbers, or for that matter any countable subset of the reals, has measure zero.

c) Describe the construction of the standard Cantor set, and show that this set has measure 0.

6. (a) Define a simple function. Define a measurable function.

(b) Suppose that f is a bounded measurable function. Given $\epsilon > 0$, show that there is a simple function ϕ such that

$$sup_{x \in R} |f(x) - \phi(x)| < \epsilon.$$

(c) Show that every bounded measurable function is the pointwise limit of an increasing sequence of simple functions.

7. a) Let f be a nonnegative measurable function defined on the real line. Show that $\int f = 0$ implies $f = 0$ almost everywhere.

b) State Fatou's lemma, the monotone convergence theorem, and the Lebesgue dominated convergence theorem.