Math 313 Fall 2006 **Quiz 1** **SOLUTION**

Show all your work. No credit will be given for answers which are not accompanied by supporting computations. Use the back of the sheet if you need more space. Circle answers when appropriate. Good luck !!

1. (2 +2 +2 = 6 pt total) In each of the three parts below, the augmented matrix of a system of linear equations is given. Describe all solutions of each system. **For any systems which have infinitely many solutions, SOLVE FOR THE LEAD VARIABLES, AND EXPRESS THE SOLUTIONS PARAMETRICALLY.** You may use the letters x, y, z, w or x_1, x_2, x_3, x_4 for variables.

(a) $\begin{bmatrix} 1 & 2 & -1 & 0 & | & 1 \\ 0 & 0 & 1 & 2 & | & -1 \end{bmatrix}$ Adding row 2 to row 1 we get $\begin{bmatrix} 1 & 2 & 0 & 2 & | & 0 \\ 0 & 0 & 1 & 2 & | & -1 \end{bmatrix}$.

Recovering the equations, $x + 2y + 2w = 0$ and $z + 2w = -1$. Let $y = r \in \mathbb{R}$ and $w = s \in \mathbb{R}$. Then the solution set is:

$$\begin{aligned} x &= -2r - 2s \\ y &= r \\ z &= -2s - 1 \qquad \text{where } r, s \in \mathbb{R}. \\ w &= s \end{aligned}$$

(b) $\begin{bmatrix} 1 & 2 & 5 & 0 & \vdots & 3 \\ 0 & 0 & 1 & 2 & \vdots & -1 \\ 0 & 0 & 0 & 0 & \vdots & 5 \end{bmatrix}$ The last row implies that $0 \cdot x + 0 \cdot y + 0 \cdot z + 0 \cdot w = 5$, that is $0 = 5$, not possible. This shows that the system has no solution. It is inconsistent.

(c) $\begin{bmatrix} 1 & 2 & 3 & | & 9 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}$ Adding -1 times row 3 to row 2 and adding -3 times row 3 row 1, we get $\begin{bmatrix} 1 & 2 & 0 & | & 3 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}$. Adding -2 times row 2 to row 3, we get $\begin{bmatrix} 1 & 0 & 0 & | & 5 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}$. Then the solution is $x = 5, y = -1, z = 2$.

2. (3 pt total) Let A be an $n \times n$ square matrix.

(a) (1 p[t) Give two different precise statements which are equivalent to the statement "A is invertible". (Remarks: **This is just asking for two of the statements from 'The Theorem'. Also, do NOT just give the definition of invertibility; that is, do not just say 'A has an inverse' or "there is a matrix B so that $AB = I$" as one of your statements**.) There are many choices.

(i) The reduced row echelon form of A is the identity matrix I_n.

(ii) The system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution.

(b)(2 pt) Suppose we are given that the system $\begin{matrix} ax + by = 3 \\ cx + dy = 4 \end{matrix}$ has only one solution. What can you conclude about the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$? Justify your answer.

M is invertible, since, as noted in Part (a) above, the system $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is given to have exacly one solution

3. (5 pt) (a) Using the elementary row operations, find the inverse of of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 0 \\ 0 & -2 & 5 \end{pmatrix}$

Applying elementary row operations successively,

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & -2 & 5 & 0 & 0 & 1 \end{array}\right)_{R_2=-R_1+R_2} \rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1 & 1 & 0 \\ 0 & -2 & 5 & 0 & 0 & 1 \end{array}\right)_{R_3=2R_2+R_3} \rightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1 & 1 & 0 \\ 0 & 0 & -1 & -2 & 2 & 1 \end{array}\right)_{R_1=-2R_2+R_1, -1\times R_3} \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 9 & 3 & -2 & 0 \\ 0 & 1 & -3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right)_{R_1=-9R_3+R_1; R_2=3R_3+}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -15 & 16 & 9 \\ 0 & 1 & 0 & 5 & -5 & -3 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right). \text{ Hence } A^{-1} = \begin{pmatrix} -15 & 16 & 9 \\ 5 & -5 & -3 \\ 2 & -2 & -1 \end{pmatrix}.$$

4. (5 pt total) (a) Find the determinant of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 3 & 7 \end{pmatrix}$ by cofactor expansion.

Expanding along the first column, $\det A = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} = 1\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} - 0 + 1\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = 1(7-6) + 0 + 1(4-3) = 1 + 1 = 2.$

5. Complete the sentence:

(a)(1 point) If A is a 2×2 matrix and $\det(A) = 4$, then $\det(2A^{-1})^T = 2^2 \cdot \det(A^{-1})^T = 4 \cdot \det(A^{-1}) = 4 \cdot 1/4 = 1$

(b) (1/2 point) A homogeneous system of 5 linear equations in 7 variable will have.....**infinitely many**..... solutions

(c) (1 point) If $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$, then $A^{-2} = (A^2)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}^{-1} = 1/4 \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} =$

$\begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}$.

(6) ($^1/2$ pt each) True / False:

(a) **F** If E is an elementary matrix, then $\det(E) = 0$ or 1. ($\det(E) = $ *0, 1 or a nonzero k*)

(b) **T** An Elementary matrix is invertible.

(c) **T** Any linear system having at least two solutions has infinitely many solutions.

(d) **F** For each $n \times n$ matrix A there is another $n \times n$ matrix B for which $AB = I_n$.(Example: $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. For any 2×2 matrix B, $AB \neq I_2$)

(e) **F** If the matrix products AB and BA are both defined, then A and B must be square matrices.(Example: A a 2×3 matrix and B a 3×2 matrix, then both AB and BA are defined.)

(f) **F** If A is a square matrix in which the entries in the main diagonal are all 0, then $\det(A) = 0$. (Example: Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\det(A) = -1 \neq 0$.)

(g) **T** For an invertible matrix A, $\det[(A^{-1})^T] = \det[(A^T)^{-1}] = \det(A^{-1})$