Review Problems - Solutions

1. Use Euler's method to find approximate values of the solution of the IVP at $t = 0.5, 1, 1.5, 2, 2.5, \text{and } 3$, with *h* = 0.05

   $y' = -ty + 0.1y^3, y(0) = 1$

   Euler's formula becomes $y_{n+1} = y_n - .05t_n y_n + .005{y_n}^3$.

t	*y*
0	1
0.5	0.9189712
1.0	0.67214519
1.5	0.36264023
2.0	0.147658886
2.5	0.04540998
3.0	0.01049305

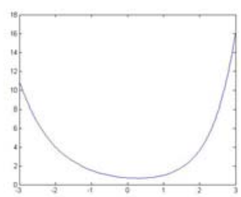
2. Solve $y'' + y' - 2y = 0$

   $y = c_1 e^{-2t} + c_2 e^t$

3. Solve $2y'' - y' - 3y = 0$, $y(1) = 1$ and $y'(1) = 1$. Sketch the graph and describe $\lim_{t\to\infty} y$

   $y = \left(\frac{4}{5}e^{-\frac{3}{2}}\right)e^{\frac{3t}{2}} + \left(\frac{1}{5}e\right)e^{-t}$

   $\lim_{t\to\infty} y = \infty$



4. Solve $y'' - 2y' + y = 0$

   $y = c_1 e^t + c_2 te^t$

5. Solve $y'' + y' + y = 0$

   $y = e^{\frac{-t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}t}{2}\right)\right)$

6. Solve $y'' + 4y = 3t\cos(2t)$

   (i) $y_h = c_1 \cos 2t + c_2 \sin 2t$

   (ii) According to the guidelines, the first guess should be $Y(t) = (At + B)\cos 2t + (Ct + D)\sin 2t$. Since $B\cos 2t + D\sin 2t$ is part of the homogeneous solution, multiply this initial guess by *t*, and check $Y(t) = (At^2 + Bt)\cos 2t + (Ct^2 + Dt)\sin 2t$. After finding $Y'(t)$ and $Y''(t)$ and substituting them into the equation, we simplify to

$(2A+4D)\cos 2t+(8C-3)t\cos 2t+(-4B+2C)\sin 2t-8At\sin 2t=0$. Thus $A=0$ and $D=0$; then $C=3/8$ and $B=3/16$.

(iii) The general solution is then: $y=c_1\cos 2t+c_2\sin 2t+\frac{3}{16}t\cos 2t+\frac{3}{8}t^2\sin 2t$

7. Given $(x-1)y''-xy'+y=0,\ x>1$ and $y_1(x)=e^x$, use the method of reduction of order to find $y_2(x)$.

   Set $y=v(x)e^x$, find y' and y'', substitute into the equation and simplify to obtain the equation: $v''=\left(-2+\frac{x}{x-1}\right)v'$, which is a separable equation in v'. Solving this for v' yields $v'=C_1(x-1)e^{-(x+1)}$. Now integrate (use a *u*-substitution and integration by parts) to find *v*: $v=-c_1xe^{-(x+1)}+c_2$. To find the second solution, multiply *v* by $y_1(x)$ to find $y=v(x)y_1(x)=-c_1xe^{-1}+c_2e^x$, which tells us that $y_2(x)=x$ (multiply the e^{-1} by the constant and call it a new constant).

8. Solve $y''-3y'=xe^{3x}$

   (i) $y_h=c_1e^{3x}+c_2$

   (ii) Initial guess is $Y(x)=(Ax+B)e^{3x}$, but this is already present in the homogeneous solution, so use $Y(x)=(Ax^2+Bx)e^{3x}$. After finding $Y'(t)$ and $Y''(t)$ and substituting them into the equation, we simplify to $(2A+3B)e^{3x}+(6A-1)xe^{3x}=0$, and $A=1/6,\ B=-1/9$.

   (iii) The general solution is $y=c_1e^{3x}+c_2+\frac{1}{6}x^2e^{3x}-\frac{1}{9}xe^{3x}$

9. Solve $y''+y=\tan x$

   (i) $y_h=c_1\cos x+c_2\sin x$

   (ii) $g(x)=\tan x, y_1(x)=\cos x,$ and $y_2(x)=\sin x$; also,

   $W(y_1,y_2)=\cos^2 x+\sin^2 x=1$, so $u_1'=\frac{-\sin x\tan x}{1}=\frac{-\sin^2 x}{\cos x}$ and

   $u_2'=\frac{\cos x\tan x}{1}=\sin x$.

   $u_1=-\int\frac{1-\cos^2 x}{\cos x}dx=\int\cos x-\sec x\,dx=\sin x-\ln|\sec x+\tan x|$

   $u_2=-\cos x$

   (iii) $Y(x)=u_1y_1+u_2y_2=-\cos x\ln|\sec x+\tan x|$, so the general solution is

   $y=c_1\cos x+c_2\sin x-\cos x\ln|\sec x+\tan x|$

10. Use variation of parameters to solve $y''+\frac{x}{1-x}y'-\frac{1}{1-x}y=1-x$, given that x and e^x are solutions to the homogeneous equation.

$W(y_1,y_2)=xe^x-e^x$, and $g(x)=1-x$, so $u_1'=\frac{-e^x(1-x)}{xe^x-e^x}=1$,

$u_2'=\frac{x(1-x)}{xe^x-e^x}=-xe^{-x}$, so $u_1=x$ and $u_2=xe^{-x}+e^{-x}$.

$y=c_1x+c_2e^x+x^2+x+1$

11. Prove that $y=t$ and $y=\ln(t)$ are linearly independent on the interval $(0,\infty)$.

The Wronskian of the two functions is $W=1-\ln t$, which is only zero at $t=e$. Thus since there is a point in the interval at which the Wronksian is nonzero, the functions are linearly independent by Theorem 3.3.1.

12. Define $L[y]$ to be the differential equation $y''+p(t)y'+q(t)y=0$. Prove that $L[y_1+y_2]=L[y_1]+L[y_2]$ and that $L[ay]=aL[y]$.

$L[y_1+y_2]=(y_1''+y_2'')+p(t)(y_1'+y_2')+q(t)(y_1+y_2)$, which simplifies to $(y_1''+p(t)y_1'+q(t)y_1)+(y_2''+p(t)y_2'+q(t)y_2)=L[y_1]+L[y_2]$. The proof of the other statement is similar.

13. Show that if $y=\varphi(t)$ is a solution of the differential equation $y''+p(t)y'+q(t)y=g(t)$, where $g(t)$ is not always zero, then $y=c\varphi(t)$, where c is a constant other than one, is not a solution.

$y=\varphi(t)$ a solution implies that $\varphi''(t)+p(t)\varphi'(t)+q(t)\varphi(t)=g(t)$. Substitution of $y=c\varphi(t)$ yields $c(\varphi''(t)+p(t)\varphi'(t)+q(t)\varphi(t))=cg(t)\neq g(t)$, if $c\neq 1$ and if $g(t)\neq 0$.

14. Prove that if y_1 and y_2 have maxima or minima at the same point in I, then they cannot be a fundamental set of solutions on that interval.

Suppose that y_1 and y_2 have maxima or minima at some point t_0 in the interval. Then $y_1'(t_0)=0$ and $y_2'(t_0)=0$, so $W(y_1(t_0),y_2(t_0))=0$. Since the Wronskian is zero at one point in the interval, the functions are not a fundamental set on that interval.

15. If the roots of the characteristic equation are real, show that a solution of $ay''+by'+cy=0$ can take on the value zero at most once. (Hint: consider the two cases with real roots separately.)

(i) Two distinct real roots. Any solution has the form $y=c_1e^{r_1t}+c_2e^{r_2t}$. By examining the graphs in the various cases (r_1 and r_2 have the same sign or different signs, c_1 and c_2 have the same sign or different signs), it is easy to see that the function can have at most one zero.

(ii) One real root. Then any solution has the form $y=c_1e^{rt}+c_2te^{rt}$. Setting this equal to zero, we have $e^{rt}(c_1+c_2t)=0$, which can have at most one solution.