Math 340
Differential Equations

Project Two

A. Introduction to Spring/Mass Systems
Read pages 192-201 (up through Example 3) in the text. Be sure you understand what a spring/mass system consists of, and what roles the spring force F_s, the damping force F_d, and the external force $F(t)$ play in the system.
(For a different presentation of the material, you might also see the attached selection from Zill's A First Course in Differential Equations with Modeling Applications).

1. A mass of 100 g stretches a spring 5cm. Suppose the mass is set in motion with a downward velocity of 10cm/sec, and that there is no damping.

   (a) What are the spring force, damping force and external force in this system? What are the values of m, k, and γ?
   (b) Write the initial value problem that governs the motion of this mass.
   (c) Solve the IVP to find the function $u(t)$ that describes the position of the mass at time t. (This is essentially problem 6 in section 3.8, so you can check your answer in the book).
   (d) Use Matlab to graph your solution function.
   One way to graph a function using Matlab is to use the `fplot` command. The syntax for this command is `fplot('function',[xmin,xmax,ymin,ymax])`. For example, the command `('exp(2*x)',[-10,5,0,1000])` plots the function $y = e^{2x}$ in the window $-10 \le x \le 5, 0 \le y \le 1000$.

   Some peculiarities about entering functions in Matlab:
   The square root function is entered as `sqrt(x)`, the natural exponential function is entered as `exp(x)`, other exponents are entered using the carat, ^, symbol as in a calculator. Operations like multiplication must be explicit. For example, enter `2*x` for 2x.

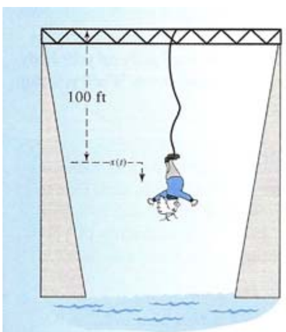
2. A 4-foot spring measures 8 feet long after a mass weighing 8 pounds is attached to it. The medium through which the mass moves offers a damping force numerically equal to $\sqrt{2}$ times the instantaneous velocity. Find the equation of motion if the mass is initially released from the equilibrium position with a downward velocity of 5 feet per second.
   (a) Write the initial value problem that governs the motion of the mass.
   (b) Solve the initial value problem and graph the solution using Matlab.

3. A force of 2 pounds stretches a spring 1 foot. A mass weighing 3.2 pounds is attached to the spring, and the system is then immersed in a medium that offers a damping force numerically equal to 0.4 times the instantaneous velocity.
   (a) Find the equation of motion if the mass is initially released from rest from a point 1 foot above the equilibrium position.
   (b) Plot your solution.

B. Bungee Jumping

Suppose that you have no sense. Suppose that you are standing on a bridge above the Malad River Canyon (near Hagerman, Idaho) and that you plan to jump off that bridge. You have no suicide wish. Instead, you plan to attach a bungee cord to your feet, to dive gracefully into the void and to be pulled back gently by the cord before you hit the river that is 174 feet below. You have brought a number of different cords to affix to your feet, including several standard bungee cords, a climbing rope, and a steel cable. You need to choose the stiffness and length of a cord to avoid the unpleasantness associated with an unexpected water landing. You are undaunted by this task, because you know differential equations!

Each of the cords you have brought will be tied off so as to be 100 feet long when hanging from the bridge. Call the position at the bottom of the cord 0, and measure the position of your feet below that "natural length" as $u(t)$, where *u* increases as you go down and is a function of time *t*. (See the figure). Then at the time you jump, $u(0) = -100$. Assuming you are 6 feet tall and you hit the water head first, then at the time you hit the water $u(t) = 174 - 100 - 6 = 68$ feet.



You know that the acceleration due to gravity is a constant *g*, so the force pulling downwards on your body is *mg*. You know that when you leap from the bridge, air resistance will increase proportionally to your speed, providing a force in the opposite direction to your motion of about β*v* where β is a constant and *v* is your velocity. Finally, you know Hooke's law describing the action of springs says that the bungee cord will eventually exert a force on you proportional of your distance past the natural length of the cord. Thus you know that the force of the cord pulling you back from destruction can be expressed as

$$b(u) = \begin{cases} 0, & u \le 0 \\ -ku & u > 0 \end{cases}$$

The number $k > 0$ in this equation is called the *spring constant* and is where the stiffness of the cord you use influences the equation. For example, if you used the steel cable, then *k* would be very large, giving a tremendous stopping force very suddenly as you passed the natural length of the cable. This could lead to discomfort, injury, or even a Darwin award. You want to choose the cord with a value of *k* large enough to stop you above or just touching the water but not too suddenly. Consequently, you are interested in finding the distance you fall below the natural length of the cord as a function of the spring constant. To do that, you must solve the differential equation that we have derived in words above: The net force mu'' on your body is given by

$$mu'' = mg + b(x) - \beta x'.$$

Here *mg* is your weight, and u' is the rate of change of your position below the equilibrium with respect to time – that is, your velocity. The constant β for air resistance depends on a number of things, including whether you wear your skin-tight pink Spandex or your skater shorts and XXL t-shirt, but you know that the value today is about 1.

The differential equation above is nonlinear (why?), but inside it are two linear equations struggling to get out. When $u < 0$, the equation is $mu'' = mg - \beta u'$, while after you pass the natural length of the cord it is $mu'' = mg - ku - \beta u'$. You will solve each equation separately and then piece together the solutions when $u(t) = 0$.

1. Solve the equation $mu'' + \beta u' = mg$ for $u(t)$, given that you *step* off the bridge – that is *no jumping, no diving*. "Stepping off" means that the initial conditions are $u(0) = -100, u'(0) = 0$. Use $mg = 160, \beta = 1,$ and $g = 32$.

2. Use the solution from Problem 1 to compute the length of time you free-fall (that is, the time it takes to go the natural length of the cord: 100 feet).

3. Compute the derivative of the solution you found in Problem 1 and evaluate it at the time you found in Problem 2. You have found your downward speed when you pass the point where the cord starts to pull.

Problem 1 has given you an expression for your position *t* seconds after you step off the bridge, before the bungee cord starts to pull you back. Notice that it does not depend on the value of *k*. When you pass the natural length of the bungee cord, it does start to pull back, so the differential equation changes. Let t_1 denote the time you computed in Problem 2, and v_1 denote the speed you calculated in Problem 3.

4. Solve the initial value problem $mu'' + \beta u' + ku = mg, u(t_1) = 0, u'(t_1) = v_1$, using the same values of *m* and β that you used above. One of your cords has a value of *k* = 16.4 – use this value for *k*. The solution $u(t)$ represents your position below the natural length of the cord after it starts to pull back.

Now you have an expression for your position as the cord pulls on your body. All you have to do is find the time t_2 at which you stop going down. When you stop going down, your velocity is zero – that is, $u'(t_2) = 0$.

5. Compute the derivative of the expression you found in Problem 4 and solve for the value of *t* where the derivative is zero. Denote this time as t_2. Be careful that the time you compute is greater than t_1 – there are several times when your motion stops at the top and bottom of your bounces! After you find t_2, substitute it back into the solution you found in Problem 4 to find your lowest position. Is this cord safe to use?

Extra Credit
You have two other cords with you, a soft bungee cord with *k* = 8.5 and a stiffer cord with *k* = 10.7. The cord with *k* = 16.4 is actually a climbing rope. Which (if any) of these three cords would be safe to use under the given conditions?

"Bungee Jumping" problems adapted from A First Course in Differential Equations with Modeling Applications by Dennis Zill