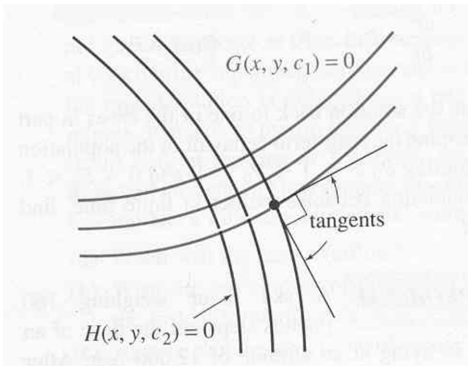
Differential Equations

Project One

1. Choose *one* of the following two questions:

A. **Orthogonal Trajectories**[1] When all curves in a family $G(x, y, c_1) = 0$ intersect orthogonally all curves in another family $H(x, y, c_2) = 0$, the families are said to be *orthogonal trajectories* of each other.




(a) Do some reading about orthogonal trajectories and their applications in the sciences, then write a short (1 – 2 page) report on your findings. Be sure you understand what you write – please do not transcribe the contents of a website for me. Also, please include a list of sources that you use.

(b) Illustrate your report by finding the orthogonal trajectories for the family $y = 1/(x + c_1)$. Use a graphing utility to graph both families on the same axes.

(c) Find the orthogonal trajectories for the family $x^2 - xy + y^2 = c_1$ (Hint: to solve the resulting D.E., see problem #30 from section 2.2, or the notes from class).

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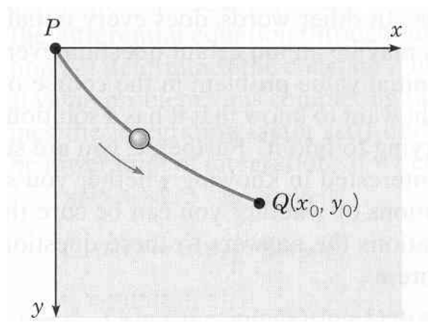
[1] Taken from *Differential Equations with Computer Lab Experiments* by Dennis G. Zill

B. **The Brachistochrone Problem**[2] One of the famous problems in the history of mathematics is the *brachistochrone problem*: to find a curve along which a particle will slide without friction in the minimum time from one given point *P* to another *Q*, the second point being lower than the first, but not directly beneath it.




(a) Do some reading about the brachistochrone problem and its historical significance. Why is this problem important still? Report your findings in a short (1 – 2 page) paper. Be sure you understand what you write – please do not transcribe the contents of a website for me. Also, please include a list of sources that you use.

(b) If we take the point *P* to be the origin, and let the lower point *Q* have coordinates $(x_0, y_0)$, the curve of minimum time is given by a function $y(x)$ that satisfies the differential equation $\left(1+\left(\frac{dy}{dx}\right)^2\right)y = k^2$, where *k* is a positive constant. Solve this differential equation for $\frac{dy}{dx}$. Why is it necessary to choose the positive square root?

(c) Introduce the new variable *t* by the relation $y = k^2 \sin^2 t$, and show that the equation found in part (b) then takes the form $2k^2 \sin^2 t \; dt = dx$.

(d) By letting $\theta = 2t$, show that the solution of the equation from part (c) for which $x = 0$ when $y = 0$ is given by

$$x = k^2(\theta - \sin\theta)/2, \quad y = k^2(1-\cos\theta)/2.$$

These are the parametric equations of the solution to the original differential equation that passes through (0,0). The graph of these equations is the *cycloid*.

(e) Finally, if we choose the constant *k* carefully, then the cycloid also passes through the point $(x_0, y_0)$ and is the solution of the brachistochrone problem. Find *k* if $x_0 = 1$ and $y_0 = 2$.

[2] Adapted from problem #32 in Section 2.3 of our book.

2. **Population Modeling**[3] If a constant number of animals *h* have been removed or harvested per unit time, then the population $P(t)$ of animals is modeled by the initial-value problem $\frac{dP}{dt} = P(a - bP) - h,\ P(0) = P_0$, where *a*, *b*, *h* and $P_0$ are positive constants.

(a) Without solving, analyze the effect of *h* on the population by finding the critical points of the equation and phase portraits in the cases $0 < h < \frac{a^2}{4b}$, $h = \frac{a^2}{4b}$, and $h > \frac{a^2}{4b}$. Where do these cases come from? Use the phase portraits to sketch the behavior of solution curves for various values of $P_0$. Use the phase portraits and solution curves to determine the long term behavior of the population. Does the population ever become extinct?

(b) Find an explicit solution of $\frac{dP}{dt} = P(5 - P) - 4,\ P(0) = P_0$ and relate the solution to one of the cases in part (a). Determine the long-term behavior of the population by considering $P_0 > 4$, $1 < P_0 < 4$, and $0 < P_0 < 1$. If the population becomes extinct in finite time, find that time.

[3] This problem is adapted from *Differential Equations with Computer Lab Experiments* by Dennis G. Zill