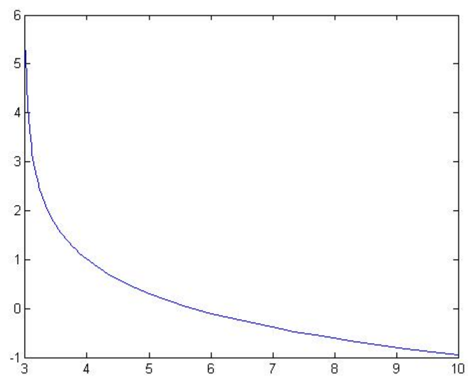
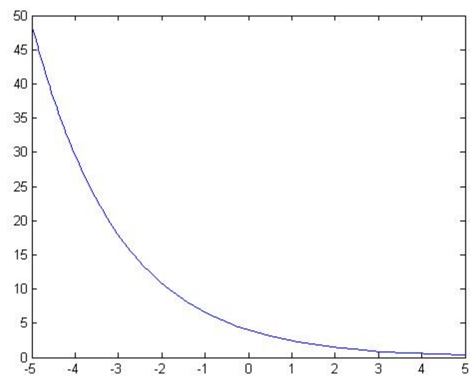
Math 135
Calculus I

Additional Practice Problems – Chapters 3 & 5
Solutions

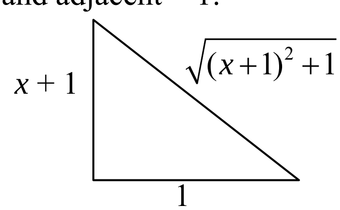
1. Sketch the graph of the following functions, using translations of a basic logarithm or exponential graph:
   a. $f(x) = 1 - \ln(x-3)$ b. $f(x) = 4e^{-x/2}$





2. Solve the equation $\ln x + \ln(x-3) = 0$ for *x*.
   $\ln(x(x-3)) = 0$
   $x^2 - 3x = 1$
   $x^2 - 3x - 1 = 0$
   Using the Quadratic Formula, we find $x = \dfrac{3 \pm \sqrt{13}}{2}$.

3. Evaluate the expression $\cos(\arctan(x+1))$ by making a sketch of a right triangle.
   Let $\theta = \arctan(x+1)$. Then $\tan\theta = x+1$, so we can set up a triangle with opposite = $x + 1$ and adjacent = 1:



$$\cos(\arctan(x+1)) = \cos\theta = \frac{1}{\sqrt{x^2+2x+2}}$$

4. Find the derivatives of the following functions:

a. $f(x) = x\sqrt{\ln x}$ $\qquad f'(x) = \sqrt{\ln x} + \frac{1}{2}(\ln x)^{-1/2} = \frac{2\ln x + 1}{2\sqrt{\ln x}}$

b. $y = 3e^{-3/t}$ $\qquad \frac{dy}{dt} = 3e^{-3/t}\left(\frac{3}{t^2}\right) = \frac{9}{t^2 e^{3/t}}$

c. $g(x) = \frac{x^2}{e^x}$ $\qquad g'(x) = \frac{2xe^x - x^2e^x}{e^{2x}} = \frac{2x - x^2}{e^x}$

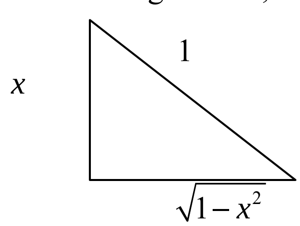
d. $y = \sqrt{e^{2x} + e^{-2x}}$ $\qquad \frac{dy}{dx} = \frac{2(e^{2x} - e^{-2x})}{2\sqrt{e^{2x} + e^{-2x}}}$

e. $y = \sqrt[5]{\frac{4x^2 - 1}{x+3}}$ $\qquad \ln y = \ln\left(\frac{4x^2-1}{x+3}\right)^{1/5} = \frac{1}{5}(\ln(4x^2 - 1) - \ln(x+3))$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{5}\left[\frac{8x}{4x^2-1} - \frac{1}{x+3}\right]$, so $\frac{dy}{dx} = \frac{1}{5}\left[\frac{8x}{4x^2-1} - \frac{1}{x+3}\right]\sqrt[5]{\frac{4x^2-1}{x+3}}$

f. $y = \frac{1}{2}\arctan e^{2x}$ $\qquad \frac{dy}{dx} = \frac{1}{1+e^{4x}}$

g. $f(x) = \tan(\arcsin x)$ $\qquad f'(x) = \sec^2(\arcsin x)\cdot\frac{1}{\sqrt{1-x^2}}$. To simplify this expression, draw a triangle. First, let $\theta = \arcsin x$. Then $\sin\theta = x = \frac{x}{1} = \frac{opp}{hyp}$, so the triangle is:



Then $\sec^2(\arcsin x) = \sec^2\theta = \frac{1}{\cos^2\theta} = \frac{1}{1-x^2}$, so $f'(x) = \frac{\frac{1}{1-x^2}}{\sqrt{1-x^2}} = \frac{1}{(1-x^2)^{3/2}}$.

5. Find $(f^{-1})'(4)$ for $f(x) = x\sqrt{x-3}$.

$(f^{-1})'(a) = \frac{1}{f'(b)}$, so first find $f'(x) = \sqrt{x-3} + \frac{x}{2\sqrt{x-3}} = \frac{3x-6}{2\sqrt{x-3}}$. Then solve the equation $f(x) = 4$ to find $b = 4$ (it's a coincidence that this is the same as the other value). Now $(f^{-1})'(4) = \frac{1}{f'(4)} = \frac{1}{3}$.

6. Evaluate the following indefinite integrals (Some of these use the technique of substitution – cancel (g) – (l)).

a. $\int(3x^4 - 6x + 5)dx = \frac{3}{5}x^5 - 3x^2 + 5x + C$ $\quad$ b. $\int 3\sqrt{x}\,dx = 2x^{3/2} + C$

c. $\int (3\cos x - 4\sin x)\,dx = 3\sin x + 4\cos x + C$ d. $\int 2\sec x \tan x\,dx = 2\sec x + C$

e. $\int \frac{3}{x^2}\,dx = \frac{-3}{x} + C$ f. $\int 5\sec^2 x\,dx = 5\tan x + C$

g. $\int t^3(1+t^4)^3\,dt$	h. $\int \cos x\sqrt{\sin x + 1}\,dx$	i. $\int \frac{6x}{(x^2-3)^2}\,dx$
j. $\int \frac{1}{16+x^2}\,dx$	k. $\int \frac{\sin x}{1+\cos x}\,dx$	l. $\int \frac{e^{1/x}}{x^2}\,dx$

7. Find the function *f*(*x*) that satisfies the conditions: $f'(x) = 3\sin x + x$, $f(0) = 4$

   $f(x) = \int (3\sin x + x)\,dx = -3\cos x + \frac{1}{2}x^2 + C$. Since $f(0) = 4$, we can substitute to find *C*: $4 = -3 + C \Rightarrow C = 7$. Thus $f(x) = -3\cos x + \frac{1}{2}x^2 + 7$.

8. Verify that the Riemann sum for $f(x) = 4 - x^2$ over the interval $[-1,1]$ is

$$\sum_{i=1}^{n}\left(4 - \left(-1 + \frac{2i}{n}\right)^2\right)\frac{2}{n}$$

   1. $\Delta x = \frac{2}{n}$ 2. $x_i = -1 + \frac{2i}{n}$ 3. $f(x_i) = 4 - \left(1 - \frac{2i}{n}\right)^2$

   Since the Riemann sum for any function is given by $\sum_{i=1}^{n} f(x_i)\Delta x$, the above sum is the Riemann sum for this function on this interval.

9. The definite integral $\int_{-1}^{1} 4 - x^2\,dx$ is defined to be $\lim_{n\to\infty}\sum_{i=1}^{n}\left(4 - \left(-1 + \frac{2i}{n}\right)^2\right)\frac{2}{n}$.

   a. Simplify the sum $\sum_{i=1}^{n}\left(4 - \left(-1 + \frac{2i}{n}\right)^2\right)\frac{2}{n}$. That is, rewrite it without the summation.

   $$\sum_{i=1}^{n}\left(4 - \left(-1 + \frac{2i}{n}\right)^2\right)\frac{2}{n} = \sum_{i=1}^{n}\left(3 + \frac{4i}{n} - \frac{4i^2}{n^2}\right)\frac{2}{n} = \sum_{i=1}^{n}\frac{6}{n} + \frac{8i}{n^2} - \frac{8i^2}{n^3}$$

   Using the summation formulas, we can rewrite this as:

   $$\frac{1}{n}\sum_{i=1}^{n}6 + \frac{8}{n^2}\sum_{i=1}^{n}i - \frac{8}{n^3}\sum_{i=1}^{n}i^2 = 6 + \frac{8}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

   b. Take the limit as $n \to \infty$.

   $$\lim_{n\to\infty} 6 + \frac{8}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = 6 + \frac{8}{2} - \frac{8}{6}\cdot\frac{2}{1} = \frac{22}{3}$$

c. Check your answer using the Fundamental Theorem of Calculus.

$$\int_{-1}^{1} 4 - x^2\, dx = 4x - \frac{1}{3}x^3 \Big|_{-1}^{1} = \left(4 - \frac{1}{3}\right) - \left(-4 + \frac{1}{3}\right) = \frac{22}{3}.$$

10. Evaluate the following definite integrals, using the Fundamental Theorem of Calculus:

a. $\int_1^2 \left(x\sqrt{x} + \frac{1}{\sqrt{x}} \right) dx = \int_1^2 x^{3/2} + x^{-1/2}\, dx = \frac{2}{5}x^{5/2} + 2x^{1/2} \Big|_1^2 = \frac{18}{5}\sqrt{2} - \frac{12}{5}$

b. $\int_1^{-1} (r+1)^2\, dr = \frac{1}{3}(r+1)^3 \Big|_1^{-1} = \frac{1}{3}\left(0^3 - 2^3\right) = -\frac{8}{3}$

c. $\int_0^1 \sqrt{t^5 + 2t}\left(5t^4 + 2\right) dt$ d. $\int_0^{\pi} x\sin(x^2)\, dx$

e. $\int_0^{\pi/8} \sec^2(2x)\, dx = \frac{1}{2}\tan(2x) \Big|_0^{\pi/8} = \frac{1}{2}\left(\tan\left(\frac{\pi}{4}\right) - \tan(0) \right) = \frac{1}{2}$

f. $\int_0^{1/2} \frac{2}{\sqrt{1-x^2}}\, dx = 2\arcsin x \Big|_0^{1/2} = 2\arcsin(1/2) - 2\arcsin(0) = 2\left(\frac{\pi}{6}\right) - 0 = \frac{\pi}{3}$

11. If oil leaks from a tank at the rate of $r(t)$ gallons per minute at time t, what does $\int_0^{120} r(t)\, dt$ represent?

It represents the total amount of oil that leaked from the tank in the first 2 hours.

12. An animal population is increasing at the rate of 200 + 5t per year (where t is measured in years). By how much does the animal population increase between the fourth and tenth years?

Amount of change = $\int_4^{10} 200 + 5t\, dt = 200t + \frac{5}{2}t^2 \Big|_4^{10} = (2000 + 250) - (800 + 40)$

= 1410 animals.

13. A population of alligators is given by the function $P(t) = \sin t + 35$, for the time period t = 0 to t = 5 (t measured in years). What is the average population over this time period?

Average value = $\frac{1}{b-a}\int_a^b f(x)\, dx$, so average population is

$$\frac{1}{5}\int_0^5 \sin t + 35\, dt = \frac{1}{5}(-\cos t + 35t) \Big|_0^5 = \frac{1}{5}((-\cos(5) + 175) - (-1 + 0)) \approx 35.14$$