Math 535 - Functional Analysis Notes 3 Morrow, Spring 2006

Bounded Linear Operators. Let X and Y be normed spaces and let $T : \mathcal{D}(T) \to Y$ be a linear operator where $\mathcal{D}(T) \subset X$. T is said to be a bounded linear operator if there exists a constant $c \geq 0$ such that

$$\|Tx\| \leq c\|x\| \text{ for all } x \in \mathcal{D}(T).$$

The norm on the left is of course the norm in Y while the norm on the right side of this inequality is the norm in X.
Exercise 2.7.2. Let $T : X \to Y$ be a linear operator. Show that T is bounded (by the above definition) if and only if T sends bounded sets in X into bounded sets in Y as follows. Suppose first that M is a bounded set in X. This means that there exists a constant $k \geq 0$ such that $\|x\| \leq k$ for every $x \in M$. Now apply T to each element of M and denote the set of images TM. Let $y = Tx$ for some $x \in M$, so $\|y\| = \|Tx\| \leq c\|x\| \leq ck$. Since this inequality must hold for every $y \in TM$ we have shown that TM is bounded. Conversely, suppose for any bounded set $M \subset X$ we have that TM is bounded in Y. Let $M = \bar{B}(0;1) \subset X$ (the closed unit ball in X). Then by assumption we have that there exists a constant $c \geq 0$ such that $\|y\| \leq c$ for every $y \in TM$. We claim that T is bounded with constant c as in the above definition. Indeed, since T is linear we have for any $x \neq 0$ that $\|Tx\| = \|x\| \cdot \|T(x/\|x\|)\| \leq \|x\|c$. QED.

The norm of a linear operator. We define the norm of a bounded linear operator as the smallest possible $c \geq 0$ such that the above definition holds, or what is the same, the supremum of $\|Tx\|/\|x\|$ over all $x \in \mathcal{D}(T)$ with $x \neq 0$. Call this quantity $\|T\|$. By the same scaling trick as shown in the last part of exercise 2.7.2, this definition of the norm of T is equivalent to

$$\|T\| = \sup_{x \in \mathcal{D}(T),\ \|x\|=1} \|Tx\|.$$

Exercise 2.7.10. Let $X = C[0,1]$ and define two linear operators S and T on X by

$$y = Sx \text{ where } y(s) = s\int_0^1 x(t)dt, \quad \text{and, } z = Tx, \text{ where } z(s) = sx(s); \quad s \in [0,1].$$

First we consider whether or not S and T commute. We have $(TSx)(s) = s(Sx)(s) = s^2 \int_0^1 x(t)dt$. Next, $(STx)(s) = s\int_0^1 (Tx)(t)dt = s\int_0^1 tx(t)dt$. These two expressions are not equal unless both are zero since the first is quadratic in s and the second is linear in s. Therefore S and T do not commute. Now we investigate the norms of these operators. First, because

$$\|y\| \leq (\max_{0 \leq s \leq 1} s) \cdot \int_0^1 |x(t)|dt \leq 1 \cdot \int_0^1 \|x\|dt = \|x\|,$$

we have that $\|S\| \leq 1$. To prove that $\|S\| = 1$, it suffices to demonstrate a function x of norm 1 such that y also has norm 1. But if x is the constant function that is equal to 1 for all t, then obviously $\|x\| = 1$, while $y = Sx$ is the function $y(s) = s$, so that also $\|y\| = \max_{0 \leq s \leq 1} s = 1$. Next it is obvious that $\|T\| \leq 1$ while by the same choice of the constant function x we also see that $|T\| \geq 1$. Further by the same argument that we used to analyze $\|S\|$, we have that $\|TS\| = 1$. Finally consider $\|ST\|$. We have the estimation

$$(\max_{0 \leq s \leq 1} s) \cdot |\int_0^1 tx(t)dt| \leq \int_0^1 t\|x\|dt = (1/2)\|x\|.$$

Therefore we have $\|ST\| \leq 1/2$. By taking again x to be the constant function 1, We find that STx is the function $y(s) = s/2$. Hence we have $\|ST\| \geq 1/2$. Thus $\|ST\| = 1/2$.

2.7-9. A Linear Operator is Bounded iff it is Continuous. Let X and Y be normed spaces and let $T : \mathcal{D}(T) \to Y$ be any operator (not necessarily linear) where $\mathcal{D}(T) \subset X$. We say that T is continuous at $x_0 \in \mathcal{D}(T)$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $\|x - x_0\| \leq \delta$, $x \in \mathcal{D}(T)$, then $\|Tx - Tx_0\| \leq \epsilon$. Also we say T is continuous if T is continuous at every $x \in \mathcal{D}(T)$. Suppose now in addition that T is linear. Then we have that
2.7-9. *(a) T is bounded if and only if T is continuous, and moreover,*
(b), if T is continuous at a single point then T is continuous.
Proof. First if T is bounded, it is easily seen to be continuous. Indeed, let $\epsilon > 0$ and choose $\delta = \epsilon/\|T\|$ independent of x_0. Here we assume $T \neq 0$ since otherwise the result is trivial. Then by boundedness and linearity, $\|x - x_0\| \leq \delta$

implies that $\|Tx - Tx_0\| = \|T(x - x_0)\| \le \|T\| \cdot \|x - x_0\| \le \delta\|T\| = \epsilon$. Thus in fact T is not only continuous but is uniformly continuous (δ may be chosen to be independent of x_0 in the definition of continuity of T at x_0). Next suppose that T is continuous at a single point x_0. Let $\epsilon = 1$. By definition there exists a $\delta > 0$ so that if $\|x - x_0\| \le \delta$, then $\|T(x - x_0)\| = \|Tx - Tx_0\| \le 1$. Now let y be any non-zero vector in $\mathcal{D}(T)$, and put $x = x_0 + \delta y/\|y\|$. Then x is correspondingly any point at distance δ from x_0. We have $\|Ty\| = \|T(x - x_0)\| \cdot \|y\|/\delta \le \|y\|/\delta$. Since y is arbitrary, we have that T is bounded with norm $\|T\| \le 1/\delta$. We have shown that continuity and linearity of T at one point implies boundedness of T, and thus by the first part of the proof, continuity of T at any point. QED

Example: linear mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$. We consider a linear mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by a 2 by 2 matrix $A = (\alpha_{j,k})_{j,k=1}^2$. Put $x = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $y = Ax = (\eta_1, \eta_2) \in \mathbb{R}^2$. Thus $\eta_1 = \alpha_{1,1}\xi_1 + \alpha_{1,2}\xi_2$, and $\eta_2 = \alpha_{2,1}\xi_1 + \alpha_{2,2}\xi_2$. If we take the Euclidean norm on $\mathbb{R}^2$ then by the Cauchy-Schwarz inequality

$$\|Ax\|^2 = \eta_1^2 + \eta_2^2 = (\alpha_{1,1}\xi_1 + \alpha_{1,2}\xi_2)^2 + (\alpha_{2,1}\xi_1 + \alpha_{2,2}\xi_2)^2 \le (\alpha_{1,1}^2 + \alpha_{1,2}^2)\|x\|^2 + (\alpha_{2,1}^2 + \alpha_{2,2}^2)\|x\|^2 = \|x\|^2 \sum_{j=1}^2 \sum_{k=1}^2 \alpha_{j,k}^2$$

Thus the linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by matrix multiplication as above satisfies $\|T\| \le \sqrt{\sum_{j=1}^2 \sum_{k=1}^2 \alpha_{j,k}^2}$. However, as indicated in Exercises 2.7.12 and 2.7.13, the norm of T is not actually equal to this last expression. This is obvious for the case of the identity matrix, wherein the norm of T is 1 but the square root of the sum of squares of the identity matrix is $\sqrt{2}$.

An alternative bound for the norm of T is obtained by the method of **2.7-8** that gives the general result that any linear operator on a finite dimensional normed space is bounded. For the case of a linear operator T from $\mathbb{R}^2$ to $\mathbb{R}^2$ as above we follow the argument to see what estimate for $\|T\|$ will arise. Put $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Then we write $x = \xi_1 e_1 + \xi_2 e_2$ and estimate that

$$\|Tx\| = \|\xi_1 Te_1 + \xi_2 Te_2\| \le |\xi_1| \cdot \|Te_1\| + |\xi_2| \cdot \|Te_2\| \le (|\xi_1| + |\xi_2|) \max\{\|Te_1\|, \|Te_2\|\}.$$

But by Lemma 2.4-1 we have that $|\xi_1| + |\xi_2| \le \|\xi_1 e_1 + \xi_2 e_2\|/c = \|x\|/c$. Here by Exercise 2.4.2(a) we have for the best constant $c = c(e_1, e_2)$ that $1/c = \sqrt{2}$, so that putting together these estimates we have $\|T\| \le \sqrt{2}\ \max\{\|Te_1\|, \|Te_2\|\} = \sqrt{2}\max\{\sqrt{\alpha_{1,1}^2 + \alpha_{2,1}^2}, \sqrt{\alpha_{1,2}^2 + \alpha_{2,2}^2}\}$. Note that this bound is always at least as large as the previous bound $\sqrt{\sum_{j=1}^2 \sum_{k=1}^2 \alpha_{j,k}^2}$.

2.7-10. *Let T be a bounded linear operator. Then the null space is closed.*
Proof. Denote the null space by $\mathcal{N}$. Let $x \in \bar{\mathcal{N}}$. We must show that $x \in \mathcal{N}$. By definition of closure there exists a sequence (x_n) in $\mathcal{N}$ such that $x_n \to x$. But since T is bounded, then by 2.7-9, T is continuous so $Tx_n \to Tx$. But $Tx_n = 0$ for every n so therefore $Tx = 0$. Hence $x \in \mathcal{N}$. QED

2.7-11. *Let $T : \mathcal{D}(T) \to Y$ be a bounded linear operator where $\mathcal{D}(T) \subset X$ and where X is a normed space and Y is a Banach space. Then T has a bounded linear extension $\tilde{T}$ to the closure of $\mathcal{D}(T)$ where in fact $\|\tilde{T}\| = \|T\|$.*
Proof. Denote the domain of T by $\mathcal{D}$. Let $x \in \bar{\mathcal{D}}$. We must show how to define $\tilde{T}x$. Let (x_n) in $\mathcal{D}$ such that $x_n \to x$. Then (x_n) is a Cauchy sequence. But since T is bounded we have that $\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \le \|T\| \cdot \|x_n - x_m\|$, so obviously the sequence of images $(y_n = Tx_n)$ is Cauchy in Y. But Y is complete, so this sequence (y_n) converges in Y to an element y. The limit y is independent of the sequence (x_n) that converges to x as follows. If (z_n) is another sequence in $\mathcal{D}$ that converges to x then we construct a sequence $(v_m) = (x_1, z_1, x_2, z_2, \dots)$. Since also (v_m) converges to x, by the proof we just gave (not by 2.7-10(a)), the limit of (Tv_m) exists in Y. Hence each of the subsequences (Tx_n) and (Tz_n) of this sequence themselves must converge to the same limit. Now it is obvious that the mapping $\tilde{T}x = y$ is linear and extends T. Finally, use that $x_n \to x$ in X implies that $\|x_n\| \to \|x\|$ (continuity of the norm) and since also $Tx_n \to \tilde{T}x$ we get likewise that $\|Tx_n\| \to \|\tilde{T}x\|$. Thus since $\|Tx_n\| \le \|T\| \cdot \|x_n\|$ is true for every n we obtain after taking the limit on $n \to \infty$ that $\|\tilde{T}x\| \le \|T\| \cdot \|x\|$. Hence $\|\tilde{T}\| = \|T\|$.

Linear Functionals. A linear functional $f : \mathcal{D}(f) \to K$ is a linear operator with a domain $\mathcal{D}(f)$ that is a subspace of a vector space X and with a range in the field of scalars K. Unless a given linear functional is the zero functional its range is in fact all of K by the scaling property $f(\alpha x) = \alpha f(x)$. Since a linear functional is a special case of a linear operator its norm $\|f\|$ is defined in the same way: $\|f\| := \sup_{x\in\mathcal{D}(f),\ \|x\|=1} |f(x)|/\|x\|$.

Examples. Let $X[0,1]$ be the space of continuous functions on $[0,1]$ with the norm $\|x\| := \sqrt{\int_0^1 x(t)^2 dt}$.
Find the norm of the linear functional defined on $X[0,1]$ by $f(x) = \int_0^1 tx(t)dt$ as follows. First, by the Cauchy-Schwarz inequality for integrals (see (1), p. 137 for $\langle x, y\rangle := \int_0^1 x(t)y(t)dt$), we have $|f(x)| \le \sqrt{\int_0^1 t^2 dt}\sqrt{\int_0^1 x(t)^2 dt} = \sqrt{1/3}\,\|x\|$. Therefore $\|f\| \le \sqrt{1/3}$. Next, if we take $x(t) = t$, $0 \le t \le 1$, then $f(x) = 1/3$ while $\|x\| = \sqrt{1/3}$. Hence $\|f\| \ge |f(x)|/\|x\| = (1/3)/\sqrt{1/3} = \sqrt{1/3}$. Thus $\|f\| = \sqrt{1/3}$.
Let f be the linear functional on $X[0,1]$ defined by $f(x) = x(0)$. Show that f is unbounded as follows. Let $n \ge 1$. Define $x_n(t)$, $t \in [0,1]$, to be a continuous function with $x_n(0) = \sqrt{n}$ and $x(t) = \sqrt{n} - (n\sqrt{n})t$, $0 \le t \le 1/n$, and $x(t) = 0$, $t \in [1/n, 1]$. It is easy to compute that $\|x_n\|^2 = \int_0^1 x_n(t)^2 dt = 1/3$. But $f(x_n) = \sqrt{n}$. Therefore for each n we have $\|f\| \ge |f(x_n)|/\|x_n\| = \sqrt{3n}$. Thus f is not bounded.
Algebraic Dual X^*. The set of linear functionals f on a vector space X is itself a vector space that we call X^*. Indeed the linear combination $\alpha f + \beta g$ with scalars α, β and linear functionals f, g on X, is again a linear functional on X. Since X^* is a vector space then the linear functionals on X^* form another vector space X^{**}, called the second algebraic dual. Note that every $x \in X$ belongs to X^{**} in the following sense. Define $g_x : X^* \to K$ for a fixed $x \in X$ by $g_x(f) := f(x)$, for all $f \in X^*$. So the values of the functional g_x on functionals f vary as f varies and x remains fixed. Obviously the element $x \in X$ may be identified with g_x. We have that the mapping $C : X \to X^{**}$ defined by $C(x) = g_x$ is linear: $C(x+y)(f) = f(x+y) = f(x) + f(y) = g_x(f) + g_y(f) = C(x)(f) + C(y)(f)$, for all $f \in X^*$, so $C(x+y) = C(x) + C(y)$. We want to know whether C is one to one (injective). By **2.6-10**, the linear operator C will be one to one if the null space is equal to the zero vector space. This is established for the case of a finite dimensional vector space X in **2.9-2** below. It can also be shown as an application of the Hahn-Banach Theorem (see **4.3-4**) that for infinite dimensional normed vector spaces X this null space is zero so that C^{-1} exists on $\mathcal{R}(C)$. In the finite dimensional case it is shown in **2.9-3** below that $\mathcal{R}(C) = X^{**}$, and so we say that X is algebraically reflexive. In section **2.10** the dual space of a normed vector space is defined as the vector space X' of all bounded linear functionals f on X, with norm $\|f\|$ as above. Then one asks (in the infinite dimensional case) whether or not a given normed space is reflexive (see **4.6**).
Dual Basis of a Finite Dimensional Vector Space. Let X be a finite dimensional vector space with basis $\{e_1, e_2, \ldots, e_n\}$. Then, represent any $x \in X$ uniquely by $x = \sum_{j=1}^n \xi_j e_j$. Thus for any linear functional $f \in X^*$ we have $f(x) = \sum_{j=1}^n \xi_j f(e_j) = \sum_{j=1}^n \xi_j \alpha_j$, for $f(e_j) = \alpha_j$, $j = 1, \ldots, n$. Thus each linear functional is determined relative to the given basis for X by an n-tuple $(\alpha_1, \ldots, \alpha_n)$. Now define a set of linear functionals $\{f_1, \ldots, f_n\}$ by the following special prescriptions for the values α_j, $j = 1, \ldots, n$. For each $k = 1, 2, \ldots, n$, put $f_k(e_j) := \delta_{j,k}$, $j = 1, \ldots, n$, where $\delta_{j,k} = 1$ if $j = k$, and $\delta_{j,k} = 0$ if $j \ne k$. Thus the representing n-tuples of these functionals are simply the standard basis vectors of $\mathbb{R}^n$, namely $(1,0,0,0,\ldots), (0,1,0,0,\ldots), \ldots$. In fact, this set of functionals gives a basis for X^*.
2.9-1 Theorem *Let X be a finite dimensional vector space with dimension n. Then the set $\{f_1, \ldots, f_n\}$ of functionals, defined above relative to a given basis of X, is a basis for X^*, so that* $\dim X = \dim X^* = n$.
Proof. Show first that $\{f_1, \ldots, f_n\}$ is a linearly independent set. If the linear combination $\sum_{k=1}^n \beta_k f_k = 0$ then for any $x \in X$ we have that $\sum_{k=1}^n \beta_k f_k(x) = 0$, so in particular by putting $x = e_j$ and by evaluating the sum using $f_k(e_j) = \delta_{j,k}$ we obtain for each $j = 1, \ldots, n$ that $\beta_j = 0$. Thus we get linear independence. To show that the set $\{f_1 \ldots, f_n\}$ spans X^* we must show that any $f \in X^*$ may be written as a linear combination. But it follows easily that if f is represented by the n-tuple $(\alpha_1, \ldots, \alpha_n)$ then in fact $f = \sum_{k=1}^n \alpha_k f_k$. Indeed with $x = \sum_{j=1}^n \xi_j e_j$ as before we have that $f_k(x) = \xi_k$. Hence $\sum_{k=1}^n \alpha_k f_k(x) = \sum_{k=1}^n \alpha_k \xi_k$, which is exactly the formula we must have for the value of $f(x)$.
Example. Let $X = \mathbb{R}^3$ and let $x = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3$. Let $e_1 = (1,1,1)$, $e_2 = (1,2,-3)$, and $e_3 = (-5,4,1)$ be a basis for X. Finally let $f(x) := \eta_1 + \eta_2 + \eta_3$ be a linear functional on X. (a) Find the representation of f in terms of the dual basis $\{f_1, f_2, f_3\}$. (b) Find $f_1(1,0,0)$.
(a) We only have to find $\alpha_j = f(e_j)$, $j = 1,2,3$. Thus $f = 3f_1 + 0f_2 + 0f_3 = 3f_1$.
(b) We have that $f_1(\sum \xi_j e_j) = \xi_1$, where $x = \sum \xi_j e_j$ is the representation of $x \in \mathbb{R}^3$ relative to the basis $\{e_1, e_2, e_3\}$. Thus we must write $x = (1,0,0)$ in terms of the given basis. This can be done as follows: $(1,0,0) = \xi_1(1,1,1) + \xi_2(1,2,-3) + \xi_3(-5,4,1)$. By solving the linear equations we find $\xi_1 = 1/3$, $\xi_2 = 1/14$, and $\xi_3 = -5/42$. Thus $f_1(1,0,0) = 1/3$.