Math 535 - Functional Analysis Notes 2 Morrow, Spring 2008

Exercise 2.3.5. **Continuity of the vector space operations (and of the norm) in a normed space.** Let $(X, \|\cdot\|)$ be a normed vector space and suppose that $x_n \to x$ and $y_n \to y$ in X and $\alpha_n \to \alpha$ in the field of scalars. Show that: (a) $x_n + y_n \to x + y$, (b) $\alpha_n x_n \to \alpha x$, and (c) $\|x_n\| \to \|x\|$.

Proof. We have by asssumption that $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$. Thus since $\|x_n + y_n - (x+y)\| \le \|x_n - x\| + \|y_n - y\|$, we have also that $\|x_n + y_n - (x+y)\| \to 0$. So (a) is proved. For part (b) we use that

$$\|\alpha_n x_n - \alpha x\| = \|\alpha_n(x_n - x) + (\alpha_n - \alpha)x\| \le |\alpha_n|\|x_n - x\| + |\alpha_n - \alpha|\|x\|.$$

Now since $|\alpha_n| \to |\alpha|$, we have by the limit theorem that $|\alpha_n|\|x_n - x\| + \|x\||\alpha_n - \alpha| \to |\alpha| \cdot 0 + 0 \cdot \|x\| = 0$. Therefore by the squeeze lemma we have that $\|\alpha_n x_n - \alpha x\| \to 0$. Finally (c) follows from the inequality (2) of Sect. 2.2-1, that gives $|(\|x_n\| - \|x\|)| \le \|x_n - x\|$.

2.4-1 Lemma. Let $\{x_1, \dots, x_n\}$ be a linearly independent set in a normed space $(X, \|\cdot\|)$ of any dimension. Then there exists a $c = c(\|\cdot\|; x_1, \dots, x_n) > 0$ such that for any scalars $\alpha_1, \dots, \alpha_n$, there holds

$$\|\alpha_1 x_1 + \dots + \alpha_n x_n\| \ge c(|\alpha_1| + \dots + |\alpha_n|). \tag{1}$$

Note. By homogeneity it is enough to prove this lemma only for scalars whose absolute values sum to 1. Indeed, if $s = |\alpha_1| + \dots + |\alpha_n| > 0$ then $|\alpha_1|/s + \dots + |\alpha_n|/s = 1$, and with $\beta_j := \alpha_j / s$ we have that (1) is equivalent to the condition that for all scalars $\beta_1, \dots, \beta_n$ with $|\beta_1| + \dots + |\beta_n| = 1$, there holds

$$\|\beta_1 x_1 + \dots + \beta_n x_n\| \ge c(|\beta_1| + \dots + |\beta_n|) = c. \tag{2}$$

Example. Let X be the set of all pairs of numbers with the norm given by $\|x\|_\infty$ of exercise 2.2.8 with $n = 2$, that is $\|x\|_\infty = \|(\xi_1, \xi_2)\|_\infty := \max\{|\xi_1|, |\xi_2|\}$. Let $x_1 = (1, 0)$ and $x_2 = (0, 1)$. Here $\{x_1, x_2\}$ is obviously a linearly independent set in X. In this example we find the largest value $\bar{c}$ of c that works in Lemma **2.4-1**. By (2), to determine the largest c we find the smallest possible value of $\|\beta_1 x_1 + \beta_2 x_2\|_\infty$ subject to $|\beta_1| + |\beta_2| = 1$. Therefore since $\beta_1 x_1 + \beta_2 x_2 = (\beta_1, \beta_2)$, the largest c is: $\bar{c} = \min_{|\beta_1| + |\beta_2| = 1} \|(\beta_1, \beta_2)\|_\infty = 1/2$. Check this: if $\bar{c} > 1/2$ then by taking the specific example $(\beta_1, \beta_2) = (1/2, 1/2)$ we find that the inequality $\|(\beta_1, \beta_2)\|_\infty \ge \bar{c}$ is false. However if $\bar{c} \le 1/2$ then for all β_1, β_2 with $|\beta_1| + |\beta_2| = 1$ we have that $\|(\beta_1, \beta_2)\|_\infty = \max\{|\beta_1|, |\beta_2|\} \ge 1/2 \ge \bar{c}$ is true. Note that the value of $\bar{c}$ we obtained here depends on the specific basis we chose for X.

Proof of Lemma 2.4-1 We show the proof only for $n = 2$ since the general case follows the same approach. Suppose by contradiction that (2) does not hold for some $c > 0$. Then there exist sequences $(\beta_1^{(m)})$ and $(\beta_2^{(m)})$ with $|\beta_1^{(m)}| + |\beta_2^{(m)}| = 1$ such that

$$\|\beta_1^{(m)} x_1 + \beta_2^{(m)} x_2\| \to 0 \text{ as } m \to \infty. \tag{3}$$

Since $|\beta_1^{(m)}| \le 1$ for all m, by compactness (of the set $|\beta| \le 1$ in $\mathbb{R}$ or $\mathbb{C}$) there is a subsequence $(\beta_1^{(m_k)})$ that converges to β_1. Now consider the corresponding subsequence of the second sequence of scalars. Again by compactness there is a subsequence $(\beta_2^{(m_{k_l})})$ that converges to β_2. Denote simply by m' the subsequnce m_{k_l} of the natural numbers so obtained. Then $(\beta_1^{(m')}, \beta_2^{(m')}) \to (\beta_1, \beta_2)$, where, since $|\beta_1^{(m')}| + |\beta_2^{(m')}| = 1$, we have that $|\beta_1| + |\beta_2| = 1$. Therefore by continuity of the vector space operations and of the norm (see Exercise 2.2.5 above) we have that $\|\beta_1^{(m')} x_1 + \beta_2^{(m')} x_2\| \to \|\beta_1 x_1 + \beta_2 x_2\| \ne 0$. the fact that this last expession is non-zero follows from the linear independence of x_1 and x_2 together with the situation that the scalars β_1 and β_2 are not both zero. But this is a contradiction since by passing to the subsequnce determined by m' we have (by our assumption "not (2)") that (3) holds also along m', so that at the same time $\|\beta_1^{(m')} x_1 + \beta_2^{(m')} x_2\| \to 0$. □

2.4-2 (Completeness) Every finite dimensional normed space X is complete.

Proof Sketch. This follows in a straightforward manner by choosing first a basis $\{x_1, \dots, x_n\}$ for X. Then consider a Cauchy sequence (y_k) in X, and write $y_k = \alpha_1^{(k)} x_1 + \dots + \alpha_n^{(k)} x_n$. Apply (1) to the representation of $y_k - y_l$ and thus conclude that, for example, the sequence $(\alpha_1^{(k)})$ is Cauchy in the field of scalars. Therefore by completeness of the scalar field one gets convergence of the coefficient sequences and so by exercise 2.3.5 above we obtain a limit in X for the sequence (y_k). □

2.4-3 (Closedness) Every finite dimensional vector subspace M of a normed space X (of any dimension) is closed.
Proof. The proof is a corollary of completeness of finite dimensional normed spaces **2.4-2**. Indeed, let $x \in \bar{M}$. By definition of the closure $\bar{M}$ there exists a sequence (x_n) in M with $x_n \to x$. But since a convergent sequence is Cauchy, and since by **2.4-2** we have that M is complete, we have that (x_n) converges to a vector $x \in M$. Therefore by uniqueness of the limit, $\bar{M} \subseteq M$. This completes the proof since therefore $M = \bar{M}$. □

Counter-example. An infinite-dimensional vector subspace Z of a normed space X need not be closed in X. Indeed let $X = C[0,1]$ and $Z = \{\text{polynomials on } [0,1]\}$. The sequence (z_n) in Z defined by $z_n(t) = 1 + t + t^2/2 + \cdots t^n/n!$, $0 \le t \le 1$, converges in $C[0,1]$ (uniform convergence) to $x(t) = e^t$, $0 \le t \le 1$. That is, $\max_{0 \le t \le 1} |x(t) - z_n(t)| \to 0$ as $n \to \infty$. Yet $x \notin Z$ since all derivatives of $x(t)$ are equal to $x(t)$, a condition that is impossible for any polynomial except zero.

2.4-4 (Equivalent norms) Let X be a finite dimensional vector space. Let $\|\cdot\|_0$ and $\|\cdot\|$ be two different norms defined on X. Then these norms are equivalent in the sense that there exist constants $a, b > 0$ so that for every $x \in X$ we have

$$a\|x\|_0 \le \|x\| \le b\|x\|_0.$$

Proof. Again we show the proof only for $\dim X = 2$. Let $\{e_1, e_2\}$ be a basis of X. Then write $x = \alpha_1 e_1 + \alpha_2 e_2$. Estimate by Lemma 2.4-1 that there is $c_0 > 0$ so that

$$\|x\|_0 / c_0 \ge |\alpha_1| + |\alpha_2|.$$

On the other hand, using the triangle inequality with the other norm we have

$$\|x\| \le |\alpha_1| \cdot \|e_1\| + |\alpha_2| \cdot \|e_2\| \le k_1(|\alpha_1| + |\alpha_2|) \text{ for } k_1 := \max\{\|e_1\|, \|e_2\|\}.$$

Therefore $\|x\|/k_1 \le |\alpha_1| + |\alpha_2|$. Hence $\|x\|/k_1 \le |\alpha_1| + |\alpha_2| \le \|x\|_0/c_0$. Thus by this last inequality $\|x\| \le (k_1/c_0)\|x\|_0$. Similarly $\|x\|_0 \le (k_0/c_1)\|x\|$ where c_1 and k_0 are determined by reversing the roles of the two norms.

Exercise 2.4.8. Let X be the vector space of pairs of numbers ($\dim X = 2$), and let $\|x\|_2 = \|(\xi_1, \xi_2)\|_2 := \sqrt{\xi_1^2 + \xi_2^2}$. Also let $\|\xi\|_1 := |\xi_1| + |\xi_2|$. Show directly that $(1/\sqrt{2})\|x\|_1 \le \|x\|_2 \le \|x\|_1$.

Proof. By squaring both sides we easily see that $\sqrt{\xi_1^2 + \xi_2^2} \le |\xi_1| + |\xi_2|$ is indeed true. Next by squaring both sides of the inequality $(1/\sqrt{2})(|\xi_1| + |\xi_2|) \le \sqrt{\xi_1^2 + \xi_2^2}$ we find after multiplyimg through by 2 and rearranging that this last inequality is equivalent to $\xi_1^2 - 2|\xi_1||\xi_2| + \xi_2^2 \ge 0$. But this last expression is the perfect square $(|\xi_1| - |\xi_2|)^2$. So we have established both norm inequalities in case$n = 2$. For the general case of n dimensions in this context the inequality $\|x\|_2 \le \|x\|_1$ follows easily as above. For the other norm inequality we may use the Cauchy-Schwarz inequality **1.2-3 (11)**. By that inequality, $\|x\|_1 = |\xi_1| \cdot 1 + \cdots + |\xi_n| \cdot 1 \le \sqrt{|\xi_1|^2 + \cdots + |\xi_n|^2}\sqrt{1^2 + \cdots + 1^2} = \sqrt{n}\|x\|_2$.

2.5-1 Compactness Definition. A metric space X is said to be compact if every sequence (x_n) in X has a subsequence that converges in X.

Note. A subset M of a metric space is again a metric space with the same metric. For example $M = [0,1) \subset \mathbb{R}$ is a metric space with the Euclidean metric. The metric subspace M is compact if it is compact as a metric space in its own right. In the example given M is not compact because the sequence $(x_n) = (1 - 1/n)$ lies in M but has no subsequence that converges in M (since the only possible limit, namely 1, does not exist in M).

2.5-2 Lemma If X is a metric space and if M is a compact subset of X then M is closed and bounded in X.

Proof. Let $x \in \bar{M}$. We want to show $x \in M$. There is a sequence (x_n) in M such that $x_n \to x$. Therefore by compactness a subsequence of this sequence converges to a point in M. But the only possible limit of such a subsequence is x itself. Therefore $x \in M$. Next we want to prove that M is bounded. Recall that M is bounded means that for some $b \in M$, $\sup_{x \in M} d(b, x) < \infty$. Now, if on the contrary M were unbounded then we could inductively construct a sequence (y_n) in M with $d(y_n, b) \ge n$. But if $y_{n_k} \to y$ then eventually, for large k, $d(y_{n_k}, b) \le 1 + d(y, b)$, which is a contradiction to our construction when $n_k \ge 1 + d(y, b)$. Therefore no subsequence of (y_n) can converge. This is a contradiction, so M is bounded.

Example. We shall see in **2.5-5** that the closed unit ball of any infinite dimensional normed space X is not compact. We can easily see this explicitly in the case $X = \ell^\infty$. Let $e_1 = (1, 0, 0, 0, \ldots)$, $e_2 = (0, 1, 0, 0, \ldots)$, etc. Then $\|e_n\| = 1$, so $e_n \in \bar{B}(0; 1)$ for every n, but also $\|e_n - e_m\| = 1$ for every $n \ne m$. Thus there is no convergent subsequence of the sequence of vectors (e_n), because no subsequence can even be a Cauchy sequence.

2.5-3 (Compactness in finite dimensions). A subset M of a finite dimensional normed space X is compact if and only if M is closed and bounded.

Proof . We assume that $\dim X < \infty$ and that M is closed and bounded. We must show that M is compact. The idea is similar to the proof in **2.4-2**. Consider for example that $\dim X = 2$. Let $\{e_1, e_2\}$ be a basis of X. Represent a given sequence (x_m) in M by $x_m = \xi_1^{(m)} e_1 + \xi_2^{(m)} e_2$. Then use Lemma **2.4-1** to write that, for some $c > 0$, independent of the choice of the sequence, $\|x_m\|/c \geq |\xi_1^{(m)}| + |\xi_2^{(m)}|$. Therefore since M is bounded so are each of the sequences of coefficients. So by compactness of the field of scalars there exists a convergent subsequence of the pairs of coefficients $(\xi_1^{(m')}, \xi_2^{(m')}) \to (\xi_1, \xi_2)$. Hence $x_{m'} \to \xi_1 e_1 + \xi_2 e_2 \in \bar{M}$. But since $M = \bar{M}$, the limit is in M. □

2.5-4 and 2.5-5 The closed unit ball in the space ℓ^∞ is not compact as shown above. It turns out that the closed unit ball of *any* infinite dimensional normed space fails to be compact. This follows by a proof employing the following Riesz's Lemma.

2.5-4 Lemma Let X be a normed space, and let Z be a subspace of X, and let Y be a proper closed subspace of Z. Then there is a vector z in the unit sphere of Z (so $\|z\| = 1$) such that $\|z - y\| \geq 1/2$ for any $y \in Y$. In fact for any fixed $0 < \theta < 1$ we can find such a z such that $\|z - y\| \geq \theta$ for any $y \in Y$.

Note. If $\dim Y < \infty$ then by Exercise 2.5.7 one may even take $\theta = 1$ in this Lemma. To see what is going on, consider $X = \mathbb{R}^3$, $Z = \{(\xi_1, \xi_2, 0) : \xi_1, \xi_2 \in \mathbb{R}\}$, and $Y = \{(\xi_1, 0, 0) : \xi_1 \in \mathbb{R}\}$. There is a vector z on the unit circle $\xi_1^2 + \xi_2^2 = 1$ in the plane Z that is at unit distance $\theta = 1$ from the line Y. Indeed there are exactly two such vectors, namely, $z = (0, \pm 1, 0)$.

2.5-5 Theorem If the closed unit ball, $M = \bar{B}(0; 1) = \{x : \|x\| \leq 1\}$, of a normed space X is compact then $\dim X < \infty$.

Proof. Assume that M is compact but $\dim X = \infty$ and obtain a contradiction by applying Riesz's Lemma repeatedly in an inductive procedure to find a sequence of vectors (x_n) on the unit sphere $\{\|x\| = 1\} \subset M$ such that $\|x_n - x_m\| \geq 1/2$ for all $n \neq m$. Here in the inductive step, $Y = \text{span}\ \{x_1, \ldots, x_n\}$, and $x_{n+1} \in Z = X$ is a unit vector that is at least a distance $1/2$ from each of the generators of Y.

Example. One may show directly that the closed unit ball $\bar{B}(0; 1)$ of $X = C[0, 1]$ is not compact directly by the following example similar in spirit to the one above for the case $X = \ell^\infty$. Indeed let $x_n \in C[0, 1]$ with $\|x\|_\infty = 1$ be defined as follows: x_n is piecewise linear and continuous with three linear pieces determined by the following data for endpoints of the line segments in its graph: $x_n(0) = 0$, $x_n(1/2^{(n+1)}) = 1$, $x_n(1/2^n) = 0$, and $x_n(1) = 0$. So the graph of $x_n(t)$, $0 \leq t \leq 1$, is a narrow "tent" of height 1 and width $1/2^n$ near the origin. By the choice of widths and heights we have that $\|x_m - x_n\|_\infty = 1$ for all $m \neq n$. Hence again there can be no subsequence of the constructed sequence that will converge.

2.6-1 Linear Operator Definition Let X and Y be vector spaces. Let also $\mathcal{D}(T)$ be a vector subspace of X such that $T : \mathcal{D}(T) \to Y$ is a mapping, denoted $y = Tx$, that satisfies:

$$T(x_1 + x_2) = Tx_1 + Tx_2 \text{ and } T(\alpha x) = \alpha Tx.$$

The set of "images" or output vectors $\mathcal{R}(T) := \{y = Tx : x \in \mathcal{D}(T)\}$ is automatically a vector subspace of Y. The null space of vectors that get sent to the zero vector, $\mathcal{N}(T) := \{x \in \mathcal{D}(T) : Tx = 0\}$, is also automatically a vector subspace of $\mathcal{D}(T)$.

Examples.

cross-product and dot product.Let $a \in \mathbb{R}^3$. Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $Tx = a \times x$ (the cross product). Also define $S : \mathbb{R}^3 \to \mathbb{R}$ by $Sx = a \cdot x$ (the dot product).

The definite integral. Define $T : C[0, 1] \to \mathbb{R}$ by $Tx = \int_0^1 x(\tau) d\tau$. For each $x \in C[0, 1]$, the element $x(t) - \int_0^1 x(\tau) d\tau \cdot 1$, $0 \leq t \leq 1$, (namely the function x minus a certain constant) belongs to the null space. Thus if we adjoin the constant function $z(t) = 1$, $0 \leq t \leq 1$, to the null space and take the span we obtain the whole domain: $\text{span}(\mathcal{N}(T), z) = C[0, 1]$.

Multiplication. Fix an element $a \in C[0, 1]$. Thus a is a continuous function $a(t)$, $0 \leq t \leq 1$. Define $T : C[0, 1] \to C[0, 1]$ by $(Tx)(t) = a(t)x(t)$, $0 \leq t \leq 1$. That is Tx is the mapping obtained by multiplying any continuous function x by the fixed continuous function a. This time if $a(t)$ has only finitely many roots, say, then the null space of T is only the zero vector because a contnuous function $x(t)$ can not have the property that it is non-zero at only finitely many points. Yet, if $a(t)$ has at least one root, then the range is not all of $C[0, 1]$ because all the image functions will have at least one root. Consider next whether T may be one to one. For example, let $a(t) = t$. Then $Tx_1 = Tx_2$ implies $t(x_1(t) - x_2(t)) = 0, 0 \leq t \leq 1$, so we must have that $x_1 = x_2$ as elements of $C[0, 1]$. Therefore T is one to one.

Note. In the previous example with $T : C[0,1] \to C[0,1]$ defined by $(Tx)(t) = tx(t)$, we showed that T is one to one. Hence T is invertible: there is a mapping $T^{-1} : \mathcal{R}(T) \to X$ defined by $T^{-1}y = y(t)/t$. Note that since y is in the range of T it must be that $y(t) = tx(t)$ for some $x \in C[0,1]$ so there can be no ambiguity in the definition of T^{-1}. Note that when T^{-1} exists it is automatically a linear operator $T^{-1} : \mathcal{R}(T) \to \mathcal{D}(T)$ with $\mathcal{R}(T^{-1}) = \mathcal{D}(T)$.

2.6-10 Inverse Linear Operator Let $T : \mathcal{D}(T) \to Y$ be a linear operator. Then (a) T^{-1} exists if and only if $\mathcal{N}(T) = \{0\}$, (b) If T^{-1} exists then T^{-1} is a linear operator, and (c) if T^{-1} exists and $\dim \mathcal{D}(T) = n < \infty$, then $\dim \mathcal{D}(T) = \dim \mathcal{R}(T)$.

Comment. Theorem **2.6-10** actually extends to the following statement known as the Rank Theorem.

Theorem. Let $T : \mathcal{D}(T) \to Y$ be a linear operator. Then $\dim \mathcal{R}(T) + \dim \mathcal{N}(T) = \dim \mathcal{D}(T)$. Here it is allowed for both sides of the equation to equal ∞. Compare Exercise 2.6.14.