Math 535 - Functional Analysis Notes 1 Morrow, Spring 2006

Vector Space. A vector space X is a set of objects that can be added together and multiplied by scalars (we consider only real or complex scalars) and with these operations the sums and scalar multiples remain in the given set of objects. The various algebraic properties of these operations including distributive laws, as well as the existence of a zero element (that we sometimes denote by θ) and an additive inverse are listed on pp. 50-51 of the text.

Examples. The Euclidean space $\mathbb{R}^n$. This is a real vector space; the scalars are real numbers.

The space $\mathbb{C}^n$ of all n-tuples of complex numbers. This is a complex vector space; the scalars are complex numbers. The scalar multiplication is defined by $\alpha(\xi_1, \xi_2, \ldots, \xi_n) = (\alpha\xi_1, \alpha\xi_2, \ldots, \alpha\xi_n)$, $\alpha \in \mathbb{C}$, $x = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{C}^n$.

Sequence space s consists of all sequences $(\xi_1, \xi_2, \xi_3, \ldots)$ of complex numbers. Addition and scalar multiplication are defined componentwise as in the case of $\mathbb{C}^n$. The zero is the sequence $(0, 0, 0, \ldots)$.

A subspace Y of a given vector space X is a non-empty subset of the elements of X that is a vector space in its own right with the inherited addition and scalar multiplication of X. By this definition, it may be proved by the axioms of a vector space (pp. 50-51) that the non-empty subset $Y \subset X$ is a subspace if and only if $\alpha_1 y_1 + \alpha_2 y_2 \in Y$ for any scalars α_1, α_2, and any elements $y_1, y_2 \in Y$. Alternatively therefore we may take this last condition as the definition of a subspace, as done on p. 53 of the text. Since $(0)x = \theta$, the zero element of X must belong to any subspace of X, and indeed the set Y consisting of only the zero element is a trivial subspace. If Y is a subspace and if x is a non-zero element of Y then the whole "line" through the origin $L := \{\alpha x : \alpha \in \mathbb{R}\}$ also belongs to Y. Thus in particular the half-plane $\{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \geq 0\}$ is **not** a subspace of $\mathbb{R}^2$.

A subspace s_0 of sequence space s is the set of sequences that have only finitely many non-zero terms in the sequence. Note that the sum of two elements of this set is still a sequence that has only finitely many non-zero terms.

Another subspace of sequence space s is the space ℓ^∞ that consists of all bounded sequences. This means that for each sequence $x = (\xi_1, \xi_2, \ldots) \in \ell^\infty$ there is a finite number $c = c_x \geq 0$ such that $|\xi_j| \leq c$ for every $j = 1, 2, 3, \ldots$ (see 1.1-6). Thus for each $x \in \ell^\infty$ we have that $\|x\| := \sup_{j \in \mathbb{N}} |\xi_j|$ (the least upper bound on the absolute values of the terms of the sequence) exists as a finite number. One can see that if $x, y \in \ell^\infty$ then both $\alpha x \in \ell^\infty$ and $x + y \in \ell^\infty$, because in fact (N3): $\|\alpha x\| = |\alpha|\|x\|$ for all scalars α, and (N4): $\|x + y\| \leq \|x\| + \|y\|$ both hold. Check this last so-called triangle inequality: $\sup_{j \in \mathbb{N}} |\xi_j + \eta_j| \leq \sup_{j \in \mathbb{N}}(|\xi_j| + |\eta_j|) \leq \sup_{j \in \mathbb{N}} |\xi_j| + \sup_{j \in \mathbb{N}} |\eta_j|$. Therefore ℓ^∞ is a vector space in its own right. This space is an example of a normed space, that is a vector space with in addition a norm defined on it, where a norm as we shall discuss at greater length below is defined in Section 2.2 of the text. Another subspace of sequence space s is defined by the set of sequences that make a different norm finite; for example the space ℓ^1 consists of sequences x such that the ℓ^1 -norm defined by: $\|x\|_1 := \sum_{j=1}^{\infty} |\xi_j|$, is finite. Again two of the defiining properties of a norm, (N3) and (N4), are easily verified in this situation, and so in particular, ℓ^1 is itself a vector space.

The space $C[a, b]$ is the set of all continuous real valued functions on $[a, b]$. Addition of two functions $x = x(t)$ and $y = y(t)$, $t \in [a, b]$, is defined by $(x + y)(t) := x(t) + y(t)$. Scalar multiplication is defined by $(\alpha x)(t) := \alpha x(t)$. $C[a, b]$ is itself a vector subspace of the space $B[a, b]$ of bounded functions on $[a, b]$. A space that sits between $C[a, b]$ and $B[a, b]$ is the set $R[a, b]$ of Riemann integrable functions on $[a, b]$. Here we use the assumption that a Riemann integrable function is bounded together with the fact that a linear combination of Riemann integrable functions is itself Riemann integrable to obtain that $R[a, b]$ is a subspace of $B[a, b]$. Also we use the fact that continuous functions are Riemann integrable to obtain that $C[a, b]$ is a subpace of $R[a, b]$.

Linear Independence. Given a vector space X and a finite collection of vectors $x_1, x_2, \ldots, x_r \in X$, we say that these vectors are linearly independent if the following condition holds: $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_r x_r = \theta$ for some scalars $\alpha_1, \alpha_2, \ldots, \alpha_r$, implies that $\alpha_1 = \alpha_2 = \cdots = \alpha_r = 0$. This is equivalent to the statement (S) that: "no one of the vectors may be written as a linear combination of the others". In fact if "not linearly independent" so that $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_r x_r = \theta$ for some scalars that are are not all zero, then we must have $x_i = -(1/\alpha_i)(\alpha_{i+1} x_1 + \alpha_{i+2} x_2 + \cdots + \alpha_r x_r)$ where α_i is the first non-zero scalar, so we have "not (S)". Conversely if "not (S)", so that one vector can be written as a linear combination of the others then the vectors are not linearly independent. Therefore we have shown "not linearly independent" if and only if "not (S)". We say that a set $M \subset X$ is linearly dependent if it is not linearly independent. Here if M is an infinite set then M is linearly independent if each finite subset of M is linearly independent.

Span and Dimension. Given a non-empty subset M of a vector space X we define a subpace of X by taking all possible linear combinations of vectors of M, and denoted the vector space by span(M). We also refer to this subspace as the subspace of X generated by M. For example, if $X = \mathbb{R}^3$, then span$\{(1,0,0),(0,1,0)\} = \{(\xi_1, \xi_2, 0) \in \mathbb{R}^3 : \xi_1, \xi_2 \in \mathbb{R}\}$. Geometrically this subspace is the plane through the origin containing the two generators $(1,0,0)$ and $(0,1,0)$. The dimension of a non-trivial vector space X is the size (cardinality-wise) of a smallest set M that generates the whole vector space, that is such that span$(M) = X$. It turns out that if there exists a finite generating set M then the size of a smallest such M is uniquely determined as a positive integer n, so the dimension is well defined as either some $n \in \mathbb{N}$ or as ∞ (the dimension of the trivial space consisting only of zero is defined by convention as the integer 0). Intuitively we have the following assertion: a smallest generating set must be linearly independent. Indeed if a generating set M is linearly dependent then some vector $x \in M$ must be expressed as a linear combination of finitely many elements of M. Thus we can take x away from M and still generate all of X. Continuing in this manner we would eliminate all dependence relations and thus be left with a linearly independent set. If there exists a finite set M that generates the space X (that is if X is finite dimensional) then this approach yields a proof of the assertion. A proof that we can always find a linearly independent set that spans or generates X when the dimension of X is infinite depends on Zorn's Lemma in Section 4.1. A linearly independent generating set M of a given vector space X is called a basis (or Hamel basis) of X (see 2.1-7). Thus we can summarize our definition of the dimension of X as the size of a basis for X. We know however that a basis itself is not unique. For example each of $\mathcal{B}_1 = \{1, t, t^2\}$ and $\mathcal{B}_2 = \{1, 1+t, 1+t+t^2\}$ is a basis for the vector space X of polynomials on $[0,1]$ that have degree at most 2. Here explicitly we write $X := \{x(t) = a_0 + a_1 t + a_2 t^2, 0 \leq t \leq 1 : a_0, a_1, a_2 \in \mathbb{R}\}$. Note: to show that $\mathcal{B}_1$ is linearly independent one can use differentiation with respect to t of the identity $\alpha_1 + \alpha_2 t + \alpha_3 t^2 = 0$, $0 < t < 1$, that appears in the condition defining linear independence of vectors. Observe that the linear independence of $\mathcal{B}_2$ will follow from the fact that span$(\mathcal{B}_2) = X$ by the uniqueness of dimension. This uniqueness is proved in the introductory course on Linear Algebra.

Normed Space. A norm on a vector space X is a real valued function on X, denoted $\|x\|$, $x \in X$, that satisfies properties (N1)-(N4) on p. 59. We have already encountered (N3) and (N4) in our discussion of subspaces in the Examples above. Property (N1) is simply that $\|x\| \geq 0$. Property (N2) states, besides $\|\theta\| = 0$, that $\|x\| = 0$ implies $x = \theta$. This last property is sometimes a little tricky to check. For example in $C[a,b]$ we may define the L^1-norm $\|x\|_1 := \int_a^b |x(t)|dt$. We verify (N2) for this definition as follows. Obviously if θ is the function that is identically zero then $\|\theta\|_1 = 0$. Next suppose that $x \in C[a,b]$ with $\|x\|_1 = 0$. We must show $x = \theta$. Suppose not. Then $\exists$ $t_0 \in [a,b]$ with $|x(t_0)| > 0$. We use the continuity of x. Let $\epsilon := |x(t_0)|/2 > 0$. Then $\exists$ $0 < \delta < (b-a)/2$ such that if $|t - t_0| < \delta$, $t \in [a,b]$, then $|x(t) - x(t_0)| < \epsilon$. This implies that $|x(t)| > |x(t_0)| - \epsilon = \epsilon$, for t belonging to an interval of length at least δ. Hence $\|x\|_1 = \int_a^b |x(t)|dt \geq \epsilon\delta > 0$, a contradiction. QED.

Complete space and Banach Space. A normed space X is a metric space that has a metric defined by $d(x,y) := \|x - y\|$, $x, y \in X$. This metric satisfies properties (M1)-(M4) on p. 3 of the text. This metric is moreover translation invariant (see 2.2-9(a)), which means that $d(x+a, y+a) = d(x,y)$ where $x + a$ and $y + a$ make sense because X is a vector space. The concepts of convergence of a sequence and of a Cauchy sequence make sense in any metric space; see 1.4-1 through 1.4-5. We review the definitions briefly here. A sequence (x_n) in X converges to $x \in X$ if $d(x_n, x) \to 0$. Also a sequence (x_n) in X is a Cauchy sequence if $\forall$ $\epsilon > 0$ $\exists$ $N \in \mathbb{N}$ such that if $m, n \geq N$ then $d(x_m, x_n) < \epsilon$. It follows immediately that any convergent sequence is a Cauchy sequence, and it also follows that any Cauchy sequence is bounded. Here the sequence (x_n) is bounded if the sequence of real numbers $(d(x_n, \theta))$ is bounded. However in a general metric space a bounded sequence need not have a subsequence that is convergent to some point in the space. If a metric space X does have the property that every bounded sequence admits a subsequence that converges to some point in X, as happens in $\mathbb{R}$ or $\mathbb{R}^n$ for example (in this case the property is known as the Bolzano-Weierstrass theorem; see appendix A1.7), then every Cauchy sequence is also convergent. Thus in $\mathbb{R}^n$ or in $\mathbb{C}^n$ we have that a sequence is convergent if and only if it is a Cauchy sequence (see 1.5-1). We say that a metric space X is **complete** if every Cauchy sequence is convergent. Thus $\mathbb{R}^n$ and $\mathbb{C}^n$ are complete normed spaces. A complete normed space is called a **Banch space**.

Examples. The normed space ℓ^∞ with the sup-norm defined above and in 2.2-4 is shown to be complete in 1.5-2.

The normed space ℓ^1 with the 1-norm defined above and also by 2.2-3 with $p = 1$ is shown to be complete in 1.5-4.

The normed space $C[a,b]$ with the norm defined by 2.2-5 (max or sup norm again) is shown to be complete in 1.5-5.

The normed space $C[a,b]$ with the L^1-norm defined above and also by 2.2-6 is shown **not** to be complete in 1.5-9.

Closed set and Closure. A subset M of a metric space X is said to be closed if (see 1.4-6(b)) the situation (x_n) in M and $x_n \to x$ implies that $x \in M$. That is M is closed means that every sequence in M that has a limit (in X) must in fact have this limit be an element of M. Here $x_n \to x$ means that (x_n) converges to x, that is $d(x_n, x) \to 0$. An accumulation point or limit point of a set M (see p. 21) is a point $x \in X$ such that there exists a sequence (x_n) in $M \setminus \{x\}$ with $x_n \to x$. For example if $X = \mathbb{R}$ and $M = \{1, 1/2, 1/3, \dots\}$, then the real number $x = 0$ is a limit point of M. In this example M has no other limit points. Since $0 \notin M$, M is not closed. Indeed a set M is closed in general if and only if it contains all its limit points. We can adjoin all the limit points to M and obtain thereby a set $\bar{M}$, called the closure of M. This is the smallest closed set containing M. See also a characterization of $\bar{M}$ in 1.4-6(a).

Example. Exercises 2.3.1 and 2.3.2. Consider the vector subspace c_0 of the Banach space ℓ^∞ that consists of real sequences that converge to zero. Since any linear combination of convergent sequences is again a convergent sequence, with the limit as the corresponding linear combination of limits (that are zero), we clealy have that c_0 is a vector subspace. We show that c_0 is closed in ℓ^∞. Indeed, let (x_n) be a sequence (of sequences) in c_0 that converges in the sup-norm to some $x \in \ell^\infty$. We must show that $x \in c_0$. Write each element $x_n = (\xi_j^{(n)}) = (\xi_1^{(n)}, \xi_2^{(n)}, \dots)$ and also $x = (\xi_j) = (\xi_1, \xi_2, \dots)$. We must show that $\lim_{j\to\infty} \xi_j = 0$. We will use by the definition of the sup-norm that $|\xi_j^{(n)} - \xi_j| \le \|x_n - x\|$ for every j. Let $\epsilon > 0$. First find N so large that $\|x_N - x\| < \epsilon/2$. Next, since $x_N \in c_0$ we have that $\lim_{j\to\infty} \xi_j^{(N)} = 0$. Hence there exists $J = J_N$ so large that if $j \ge J$ then $|\xi_j^N| < \epsilon/2$. Therefore if $j \ge J$ we have by the triangle inequality for real numbers that $|\xi_j| \le |\xi_j - \xi_j^{(N)}| + |\xi_j^{(N)}| \le \|x - x_N\| + |\xi_j^{(N)}| < \epsilon/2 + \epsilon/2 = \epsilon$. Therefore we have established that $\lim_{j\to\infty} \xi_j = 0$. QED

Complete subspace. A subset M of a complete metric space X is itself a complete metric space with the inherited metric if and only if M is closed in X (Theorem 1.4-7). Therefore a vector subspace Y of a Banach space X is itself a Banach space (that is a complete normed space with the norm inherited from X) if and only if Y is closed in X (again, relative to the metric $d(x, y) = \|x - y\|$). Therefore we have shown that the normed space c_0 with the sup-norm of the previous example is a Banach space because c_0 is a closed subspace of the Banch space ℓ^∞.

Completion. If a normed space X is not complete then there is a general procedure for completing the space to a larger space shown in 1.6 and 2.3-2. The larger space $\hat{X}$ consists of elements that are equivalence classes of Cauchy sequences from X, where two Cauchy sequences (x_n) and (x'_n) are equivalent if $d(x_n, x'_n) \to 0$. The Cauchy sequences that converge to a given element $x \in X$ are all in one equivalence class, and this is simply another way of writing the element x. But there are Cauchy sequences that don't converge to any element of X. The distance between two equivalence classes $\hat{x}$ and $\hat{y}$ with representatives (as Cauchy sequences) (x_n) and (y_n) respectively is $\hat{d}(\hat{x}, \hat{y}) = \lim_{n\to\infty} d(x_n, y_n)$ (the limit exists by the argument on p. 42). Obviously if $\hat{x}$ and $\hat{y}$ belong to X to begin with in the sense that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\hat{d}(\hat{x}, \hat{y}) = d(x, y) = \|x - y\|$. As shown in 1.6-2 and 2.3-2, the space $\hat{X}$ is a Banach space with norm $\|\hat{x}\|_2 = \hat{d}(\theta, \hat{x})$. Further, the space X is dense in $\hat{X}$ as follows. For every $\epsilon > 0$ and for every $\hat{x} \in \hat{X}$, there is some $x \in X$ with $\hat{d}(x, \hat{x}) < \epsilon$. Kreyszig notes in 2.2-7 that the completion of $C[a, b]$ with the L^1-norm (given by (8) in 2,.2-7 with $p = 1$) is in this way a Banach space called $L^1[a, b]$. This completion can be done for each of the L^p norms, $1 \le p < \infty$, defined also by formula (8) of 2.2-7.