Math 535 - Modern Analysis II Notes-2 Morrow, Spring 2006

Uniform Convergence A sequence of functions $\{f_n : D \to \mathbb{R}\}$ converges uniformly to a function $f : D \to \mathbb{R}$ if the follwing condition holds. For every $\epsilon > 0$ there exists a positive integer $N = N_\epsilon$ such that

$$\sup_{x \in D} |f_n(x) - f(x)| < \epsilon$$

This condition is equivalent to the so called uniform Cauchy condition (p. 209) as far as the existence of uniform convergence to a limit function f goes (Theorem 9.16, Weierstrass uniform convergence criterion). The difference in the two conditions is that in the Cauchy condition there is no explict mention of the limit function f even though this limit necessarily exists under the condition. The uniform Cauchy condition is stated: for every $\epsilon > 0$ there is a a positive integer $N = N_\epsilon$ such that for all $m, n \geq N$ there holds

$$\sup_{x \in D} |f_n(x) - f_m(x)| < \epsilon$$

Examples. We give some examples and methods for establishing uniform convergence and non-uniform convergence. Example 1. For each postive integer $n \geq 2$, let $f_n : [0,1] \to \mathbb{R}$ be defined as a piecewise linear and continuous function such that $f_n(0) = 0$, $f(x) = 0$, $2/n \leq x \leq 1$, and f_n is linear on each of the intervals $[0, 1/n]$ and $[1/n, 2/n]$ with $f_n(1/n) = 1$. The region under the graph of f_n and above the x-axis is the region inside an isoceles triangle with base the interval $[0, 2/n]$ and vertex the point $(1/n, 1)$ in the plane. It is shown in the handwritten Notes-1 that $\{f_n\}$ converges pointwise on $[0,1]$ to the constant function $f(x) = 0$, $x \in [0,1]$. Note that the first definition above yields automatically that if the sequence is uniformly convergent to f then it is pointwise convergent with the same limit f. Thus we must take $f(x) = 0$, $0 \leq x \leq 1$, to check in this example whether the uniform convergence holds or not. Now we simply compute that $\sup_{x \in [0,1]} |f_n(x) - f(x)| = 1$ The supremum is in fact a maximum since both f_n and f are continuous in this example, and the maximum difference occurs at $x = 1/n$. Thus if $0 < \epsilon < 1$ then in particular for any cut-off index N, not only is there some $n \geq N$ but in fact for all n we have that $\sup_{x \in [0,1]} |f_n(x) - f(x)| > \epsilon$. Geometrically, the graph of f_n does not lie in the ϵ-tube about the graph of f for any $\epsilon < 1$ and any $n \geq 2$.

Example 2.Let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = nx^n(1-x)$. We first establish that $\{f_n\}$ converges pointwise on $[0,1]$. First, if $x = 1$ then $f_n(1) = 0$ for every n so $\lim_{n\to\infty} f_n(1) = 0$. Likewise if $x = 0$ then $\lim_{n\to\infty} f_n(0) = 0$. To establish the limit in case $0 < x < 1$ we apply the following lemma whose proof is a corollary of the ratio test 9.15 and the divergence test 9.5.

Lemma 1 *Let $\{a_n\}$ be a sequence of positive numbers such that $\lim_{n\to\infty} a_{n+1}/a_n = r$. If $r < 1$ then $\lim_{n\to\infty} a_n = 0$. If $r > 1$ then the sequence $\{a_n\}$ diverges. No information about convergence or divergence may be extracted if $r = 1$.*

Now fix $0 < x < 1$. We take $a_n = nx^n(1-x)$. Then

$$\lim_{n\to\infty} a_{n+1}/a_n = \lim_{n\to\infty} (n+1)x^{n+1}/(nx^n) = x \lim_{n\to\infty} (n+1)/n = x < 1.$$

Thus by Lemma 1 we have that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f_n(x) = 0$, $0 < x < 1$. Hence by this and our preceding observations, we have that a pointwise limit exists and is given by $f(x) = 0$, $x \in [0,1]$. We now attempt to calculate or estimate $d_n := \sup_{x\in[0,1]} |f_n(x) - f(x)|$ (this is the distance between f_n and f in the space $C([0,1], \mathbb{R})$ of Thm. 12.3, p. 272.). We know, since $f(x) = 0$ and since the function $f_n(x)$ is zero at the endpoints, continuous on $[0,1]$ and differentiable on $(0,1)$, that this supremum is actually a maximum that must occur at a critical point in $(0,1)$. We have $f_n'(x) = (nx^n - nx^{n+1})' = nx^{n-1}(n - (n+1)x)$. Thus $f_n'(x) = 0$ if and only if $x = n/(n+1)$ or $x = 0$. Thus the supremum is calculated as $d_n = f_n(n/(n+1)) = n(n/(n+1))^n(1/(n+1)) = (n/(n+1))^{n+1}$. Finally, compute that

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} e^{\ln d_n} = \lim_{n\to\infty} e^{(n+1)\ln(n/(n+1))}$$

We can compute the limit in the exponent by L'Hopital's rule as

$$\lim_{t\to\infty} \ln(t/(t+1))/(1/(t+1)) = \lim_{t\to\infty} (\ln(t) - \ln(t+1))'/(1/(t+1))' = -\lim_{t\to\infty} (1/t - 1/(t+1))(t+1)^2 = -\lim_{t\to\infty} (t+1)/t = -1$$

Therefore $\lim_{n\to\infty} d_n = e^{-1}$. Now if $\lim_{n\to\infty} d_n = 0$ then the convergence is uniform, and otherwise it is not. Hence the sequnce $\{f_n\}$ does not converge uniformly on $[0,1]$. For the same reason the sequence does not converge uniformly on $[0,1)$. However, it may be shown that for any $0 < a < 1$, the sequence does converge uniformly on $[0,a]$.

Example 3. Let $f_n(x) = \sqrt{x+1/n}$, $x \geq 0$. First fix $x \geq 0$ and put $u_n := x+1/n$. Obviously $u_n \to x$, as $x \to \infty$, so by continuity of $f(u) = \sqrt{u}$ at $u = x$ we have that $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f(u_n) = f(x) = \sqrt{x}$. Thus we have a pointwise limit. To determine whether or not there is uniform convergence, we calculate $d_n := \sup_{x\geq 0} |\sqrt{x+1/n} - \sqrt{x}\,|$ and see if $d_n \to 0$ as $n \to \infty$ or not. We have

$$\sqrt{x+1/n} - \sqrt{x} = \frac{(\sqrt{x+1/n} - \sqrt{x})(\sqrt{x+1/n} + \sqrt{x})}{\sqrt{x+1/n} + \sqrt{x}} = \frac{1/n}{\sqrt{x+1/n} + \sqrt{x}}$$

Notice that the denominator is smallest among all $x \geq 0$ when $x = 0$. Therefore the supremum is actually attained at $x = 0$ and is equal to $d_n = (1/n)/(1/\sqrt{n}) = 1/\sqrt{n}$. Since $\lim_{n\to\infty} d_n = 0$ we have uniform convergence.

Example 4. Let $0 < a < 1$ and define $f_n(x) := \sum_{j=1}^{n} n^2 x^n$, $x \in [-a,a]$. Show using Thm. 9.16 that $\{f_n : [-a,a] \to \mathbb{R}\}$ converges uniformly. Since we will use the Cauchy criterion, we avoid an explicit calculation of the limit $f(x)$. Instead we shall estimate $e_n := \sup_{x\in[-a,a]} \sup_{k\in\mathbb{N}} |f_n(x) - f_{n+k}(x)|$ (the order of the sup's may be interchanged without changing the value of e_n). We know that uniform convergence holds if and only if $e_n \to 0$ as $n \to \infty$. Our plan is to find an upper bound for e_n that itself converges to zero. Indeed, we have

$$|f_{n+k}(x) - f_n(x)| = |\sum_{j=n+1}^{n+k} j^2 x^j| \leq \sum_{j=n+1}^{n+k} j^2 a^j = \sum_{j=n+1}^{n+k} M_j \leq \sum_{j=n+1}^{\infty} M_j$$

for $M_j = j^2 a^j \geq 0$. Notice that this estimation gives a bound independent of both x and k. Now suppose that $\sum_{j=1}^{\infty} M_j$ is convergent. Then we know that the tail of the series $\sum_{j=n+1}^{\infty} M_j$, that itself serves as an upper bound for e_n, converges to zero. Hence by the squeezing principle, e_n converges to zero. To verfify that $\sum_{j=1}^{\infty} M_j$ converges we apply the ratio test (Thm 9.15) and easily obtain $\lim_{n\to\infty} M_{j+1}/M_j = a < 1$. Thus we have uniform convergence.

The outline of the proof in the last example is summarized in the following Theorem, commonly called the Weierstrass M-test.

Theorem 1 *Let $\sum_{j=1}^{\infty} a_j(x)$ be a series of real valued functions on a domain D. Assume that there exist constants $M_j \geq 0$ such that*
(1) $|a_j(x)| \leq M_j$ (uniformly) for all $x \in D$, and
(2) $\sum_{j=1}^{\infty} M_j$ is a convergenet series.
Then $\{f_n(x) := \sum_{j=1}^{n} a_j(x) : D \to \mathbb{R}\}$ is a uniformly convergent sequence of functions. We say that the series $\sum_{j=1}^{\infty} a_j(x)$ converges uniformly (and absolutely) on D.

Example 4a. Suppose in Example 4 that we take the domain to be $(-1,1)$ instead of $[-a,a]$. We still have pointwise convergence of the series $\sum_{j=1}^{\infty} j^2 x^j$ since each x satisfies $|x| < 1$. However, for any n we have that $\sup_{x\in(-1,1)} |f_n(x) - f_{n+1}(x)| = \sup_{x\in(-1,1)} |(n+1)^2 x^{n+1}| = (n+1)^2$. In fact

$$e_n := \sup_{x\in(-1,1)} \sup_{k\in\mathbb{N}} |f_n(x) - f_{n+k}(x)| = \sup_{k\in\mathbb{N}} \sup_{x\in(-1,1)} |\sum_{j=n+1}^{n+k} j^2 x^j| = \infty.$$

Thus e_n does not converge to zero, so there is no uniform convergence.