

Math 532 Modern Analysis II Review Test 2 Morrow, Spring 2006

Use of any theorem, example, or exercise through Chapter 12 of the text is permitted.

1. (a) Define $A := \{(x, \tan(x)) \in \mathbb{R}^2 : -\pi/2 < x < \pi/2\}$. Prove that A is closed in \mathbb{R}^2 .
(b) Define $B := \{(x, \sin(x)) \in \mathbb{R}^2 : -\pi/2 < x < \pi/2\}$. Prove that B is neither closed nor open in \mathbb{R}^2 .

2. Let $S := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.
(a) Define $q : S \rightarrow \mathbb{R}$ by $q(x, y) = 4x^2 - 3xy + 8y^2 + 1/(x - 2) + 1/(y - 3)$. Prove that if $\{(x_k, y_k)\}$ is a sequence of points in S , then for some subsequence $k_1 < k_2 < \dots$ of the positive integers we have that the sequence $\{q(x_{k_l}, y_{k_l})\}$, $l = 1, 2, 3, \dots$, converges in \mathbb{R} .
(b) Let $p(x, y) = ax^2 + bxy + cy^2$ for some constants a, b, c . Assume that $p(x, y) > 0$ for all $(x, y) \neq (0, 0)$. Prove that: $\inf_{(x,y) \in S} p(x, y) > 0$.

3. Let $X = C([0, 1], \mathbb{R})$ with the uniform metric.
(a) Let $A := \{f \in X : \int_0^1 f(x) \sin(\pi x) dx = 0\}$.
Show that (i) A is closed in X , and (ii) A is not open in X .
(b) Suppose that $\{f_n\}$ is a sequence in X with the properties that: (I) $\max_{0 \leq x \leq 1} |f_n(x)| = 1$ for each $n = 1, 2, 3, \dots$ and (II) for each fixed $x \in [0, 1]$, $\lim_{n \rightarrow \infty} f_n(x) = 0$. Prove that $\{f_n\}$ has no convergent subsequence in X .

4. (a) Let a sequence of real numbers $\{x_n\}$ be defined by $x_1 = 0$ and $x_{n+1} = \ln(3 + \cos(x_n))$ for $n \geq 1$. Prove that the sequence $\{x_n\}$ converges in \mathbb{R} to some limit L .
(b) Suppose instead that $x_1 = 1$ and define again $x_{n+1} = \ln(3 + \cos(x_n))$ for $n \geq 1$. Prove that $\lim_{n \rightarrow \infty} x_n = L$ for the same L as in part (a).

5. Let $X = C([0, 1], \mathbb{R})$ with the uniform metric. Define $\Phi : X \rightarrow X$ by $\Phi(f) = h$ for $h(x) = (x + f(x^2))/2$, $x \in [0, 1]$. Prove that Φ is a contraction on X . Conclude that Φ has a fixed point. Find this fixed point as a convergent power series for extra credit.

6. Let (X, d) be a metric space and suppose that $f : [0, 1] \rightarrow X$ is continuous. Prove that $A := f([0, 1])$ is closed in X .