Math 532 Modern Analysis II Review for Final Exam Morrow, Spring 2006

Final Exam: May 9.

1. Show that

$$\text{(a)} \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} \quad \text{does not exist, and (b) (Extra Credit)} \lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^4} = 0$$

2. Prove that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)-(x+y)}{x^2+y^2} = 0$$

3. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by $f(x,y)=(x-1)^2+(x+y)^2+(x+2y-3)^2+(x+3y-2)^2$.
(a) Prove that f has a unique critical point $\mathbf{x}_0=(x_0,y_0)$ and that f attains a local minimum value $f=3.2$ at that point.
(b) Let $\mathbf{x}_0$ be the critical point from part (a). Write out the formula

$$f(\mathbf{x}_0+\mathbf{h}) = f(\mathbf{x}_0) + \langle\nabla f(\mathbf{x}_0),\mathbf{h}\rangle + (1/2)\langle\nabla^2 f(\mathbf{x}_0+\theta\mathbf{h})\mathbf{h},\mathbf{h}\rangle$$

explicitly in terms of $\mathbf{h}=(h_1,h_2)$. Here $\theta\in(0,1)$.

4. Suppose that $f:\mathbb{R}^2\to\mathbb{R}$ is such that the first order partial derivatives exist everywhere.
(a) Assume that $\frac{\partial f}{\partial x}(x,y)=0$ and $\frac{\partial f}{\partial y}(x,y)=0$ for all $(x,y)\in\mathbb{R}^2$. Prove that $f(x,y)=f(0,0)$ for all $(x,y)\in\mathbb{R}^2$.
(b) Assume instead that $\frac{\partial f}{\partial x}(x,y)=a$ and $\frac{\partial f}{\partial y}(x,y)=b$ for all $(x,y)\in\mathbb{R}^2$, where a and b are real constants. Prove that $f(x,y)=f(0,0)+ax+by$ for all $(x,y)\in\mathbb{R}^2$.

5. Suppose that $f:\mathbb{R}^3\to\mathbb{R}$ has continuous second order partial derivatives on $\mathbb{R}^3$. Suppose that $\gamma:\mathbb{R}\to\mathbb{R}^3$ is differentiable to all orders. Define $g:\mathbb{R}\to\mathbb{R}$ as the composition $g=f\circ\gamma$, that is, $g(t)=f(\gamma(t))$, $t\in\mathbb{R}$.
(a) Let γ be the linear mapping $\gamma(t)=t\mathbf{u}$ for a fixed vector $\mathbf{u}\in\mathbb{R}^3$, so that $g(t)=f(t\mathbf{u})$, $t\in\mathbb{R}$. Use the chain rule to show directly that

$$\text{(i) } g'(0)=\langle\nabla f(\mathbf{0}),\mathbf{u}\rangle, \text{ and (ii) } g''(0)=\left\langle \mathbf{D}^2 f(\mathbf{0})\mathbf{u},\mathbf{u}\right\rangle$$

(b) Suppose instead that $\gamma(t)=t\mathbf{u}+t^2\mathbf{v}$ for fixed vectors $\mathbf{u}\in\mathbb{R}^n$ and $\mathbf{v}\in\mathbb{R}^n$, so that $g(t)=f(t\mathbf{u}+t^2\mathbf{v})$, $t\in\mathbb{R}$. Show that

$$\text{(i) } g'(0)=\langle\nabla f(\mathbf{0}),\mathbf{u}\rangle, \text{ and (ii) } g''(0)=\left\langle \mathbf{D}^2 f(\mathbf{0})\mathbf{u},\mathbf{u}\right\rangle + 2\langle\nabla f(\mathbf{0}),\mathbf{v}\rangle$$