Math 432/532 Modern Analysis II Notes Feb. 22-24 Morrow, Spring 2005

Continuity of a function F from a set $A \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$.
Let $A \subseteq \mathbb{R}^n$ and let $F : A \to \mathbb{R}^m$. Let $\mathbf{u} \in A$. We say that F is continuous at $\mathbf{u}$ if whenever $\{\mathbf{u}_k\}$ is a sequence of points in A with limit $\mathbf{u}$ then also the sequence $\{F(\mathbf{u}_k)\}$ converges in $\mathbb{R}^m$ to the point $F(\mathbf{u})$. We say $F : A \to \mathbb{R}^m$ is continuous if F is continuous at each point of A.

Consider $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x, y) = (x^2 - y^2, 2xy)$$

Alternatively we write

$$F_1 = x^2 - y^2, F_2 = 2xy, \text{ where } F = (F_1, F_2)$$

We claim that F is continuous at any point $(x, y) \in \mathbb{R}^2$. To verify this, let $\{(x_k, y_k)\} \to (x, y)$ in $\mathbb{R}^2$. By componentwise convergence (Thm. 10.11) we know that $\{x_k\} \to x$ and $\{y_k\} \to y$. Therefore by the product and sum limit theorems for real sequences we have $\{x_k^2 - y_k^2\} \to x^2 - y^2$ and $\{2x_k y_k\} \to 2xy$. Thus again by componentwise convergence we know that $\{(x_k^2 - y_k^2, 2x_k y_k)\} \to (x^2 - y^2, 2xy)$ in $\mathbb{R}^2$. QED

By the above example we know that polynomials $p : \mathbb{R}^n \to \mathbb{R}$ are continuous, and quotients of such polynomials (rational functions in n -variables) are continuous at all points in $\mathbb{R}^n$ where the denominator doesn't vanish. Also, a given function $F = (F_1, F_2, \ldots, F_m) : A \to \mathbb{R}^m$ such that each component F_i is a real valued continuous function at a point $\mathbf{u} \in A \subseteq \mathbb{R}^n$ is also continuous at $\mathbf{u}$. Using the same ideas one obtains Corollary 11.15: a linear combination of continuous mappings is continuous.

Composition.
Theorem 11.3 on composition of contnuous mappings follows from the definiton of contnuity above by directly checking the condition that the composition evaluated along a sequence of points in the domain of the composition converges appropriately. See the statement and proof of this theorem, page 251 of the text.

Example. Let $F = (F_1, F_2, F_3) : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$F_1 = \sin(e^{xz} + x^2 - xy), F_2 = \arctan(xz - \cos(xy)), F_3 = x^2 + y^2 + z^2$$

Then each of F_1, F_2, and F_3 are real valued continuous functions by composition using Theorem 11.3. Therefore by Theorem 11.4 (componentwise continuity)the mapping F is continuous. To see the continuity of F_1 explicitly we write $F_1 = \sin(\exp(p_1 \cdot p_3) + p_1^2 - p_1 \cdot p_2)$.

The ϵ - δ definition of continuity.
Theorem 11.6 gives the states the equivalence of the above definition of continuity with the following ϵ - δ version. Let $A \subseteq \mathbb{R}^n$ and let $F : A \to \mathbb{R}^m$. Let $\mathbf{u} \in A$. F is continuous if and only if for every $\epsilon > 0$ there eists a $\delta > 0$ such that if $d(\mathbf{v}, \mathbf{u}) < \delta$ then $d(F(\mathbf{v}), F(\mathbf{u})) < \epsilon$.

Example. Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. Define

$$A := \{\mathbf{u} \in \mathbb{R}^n : f(\mathbf{u}) > 0\}$$

We claim that A is open in $\mathbb{R}^n$. Indeed, let $\mathbf{u} \in A$. Then $f(\mathbf{u}) = \epsilon > 0$. By the ϵ - δ definition of continuity, there exists $\delta > 0$ such that if $d(\mathbf{v}, \mathbf{u}) < \delta$ then $|f(\mathbf{v}) - f(\mathbf{u})| < \epsilon$. Therefore,

$$f(\mathbf{v}) = f(\mathbf{u}) - (f(\mathbf{u}) - f(\mathbf{v})) > \epsilon - \epsilon = 0$$

Hence $\mathbf{v} \in A$. Therefore the whole symmetric neighborhood of radius δ about $\mathbf{u}$ belongs to A. Since $\mathbf{u}$ was an arbitrary point in A, we have that A is open in $\mathbb{R}^n$.

Another characterization of continuity.
Let O be an open subset of $\mathbb{R}^n$. Theorem 11.7 states that $F : O \to \mathbb{R}^m$ is continuous if and only if for each open set $V \subseteq \mathbb{R}^m$ we have that $f^{-1}(V)$ is an open subset of $\mathbb{R}^n$. The proof is similar to the solution of the last example.
Example. Exercise 11.1 # 11. Let O be an open subset of $\mathbb{R}^n$ and assume that $f : O \to \mathbb{R}$ is continuous. Let $\mathbf{u} \in O$ and suppose that $f(\mathbf{u}) > 0$. Prove that there is a symmetric neighborhood N of $\mathbf{u}$ such that $f(\mathbf{v}) > f(\mathbf{u})/2$ for all $\mathbf{v} \in N$. One proof is to denote $r := f(\mathbf{u}) > 0$. Then take $V = (r/2, 3r/2) \subseteq \mathbb{R}$. We have that $f^{-1}(V)$ is open and contains $\mathbf{u}$. Therefore there exists a symmetric neighborhood $N = N_\delta(\mathbf{u}) \subseteq \mathbb{R}^n$ such that

$$f(N) \subseteq (r/2, 3r/2) \subseteq \mathbb{R}$$

hence for every $\mathbf{v} \in N$ we have $f(\mathbf{v}) > r/2$. QED