Math 532 Modern Analysis II Notes 10 Morrow, Spring 2006

Directional Derivative.

Example. Let $\mathcal{O} \subseteq \mathbb{R}^2$, and let $f : \mathcal{O} \to \mathbb{R}$. We first sketch the Mean Value Proposition 13.7. Assume that the first partials of f exist on $\mathcal{O}$. Let $\mathbf{x} = (x, y) \in \mathcal{O}$ and let $r > 0$ be such that $\mathcal{N}_r((x, y)) \subseteq \mathcal{O}$. Then let $\mathbf{x} + \mathbf{h} = (x + h_1, y + h_2) \in \mathcal{N}_r((x, y))$. Then there exist $\theta_1, \theta_2 \in (0, 1)$ such that

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = h_1 \frac{\partial \mathrm{f}}{\partial \mathrm{x}}(x + \theta_1 h_1, y + h_2) + h_2 \frac{\partial \mathrm{f}}{\partial \mathrm{y}}(x, y + \theta_2 h_2)$$

The proof follows by simply applying the Lagrange mean value theorem to a differentiable function of one variable. Indeed, write

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = [f(x + h_1, y + h_2) - f(x, y + h_2)] + [f(x, y + h_2) - f(x, y)]$$

simply apply the mean value theorem to the first difference in square brackets treated as a function of the single variable x with the second variable fixed at $y + h_2$. Similarly view the second difference as a difference of a function of the single variable y. QED.

The Directional Derivative is defined at $\mathbf{x} = (x, y) \in \mathcal{O}$ by $\frac{\partial \mathrm{f}}{\partial \mathbf{p}}(\mathbf{x}) = \lim_{t\to 0}(f(\mathbf{x} + t\mathbf{p}) - f(\mathbf{x}))/t$ for any non-zero vector $\mathbf{p} \in \mathbb{R}^2$ (see text p. 322). Usually we consider only $\mathbf{p}$ as a unit vector since then the directional derivative may be interpreted as the instantaneous rate of change of f with respect to changes in dstance in the plane in the direction of $\mathbf{p}$ at the point $\mathbf{x}$. Therefore if $\|\mathbf{p}\| = 1$ we may write the directional derivative instead as $\frac{\mathrm{df}}{\mathrm{ds}}$ where s stands for distance in the plane. The theorem 13.8 states that if $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable (has continuous first order partials in $\mathcal{O}$) then the directional derivative exists at each point $\mathbf{x} \in \mathcal{O}$ and each $\mathbf{p} \neq \mathbf{0}$. Moreover the formula for this directional derivative is

$$\frac{\partial \mathrm{f}}{\partial \mathbf{p}}(\mathbf{x}) = p_1 \frac{\partial \mathrm{f}}{\partial \mathrm{x}}(x, y) + p_2 \frac{\partial \mathrm{f}}{\partial \mathrm{y}}(x, y)$$

This follow immediately from the Mean value Proposition 13.7 with $\mathbf{h} = t\mathbf{p}$ together with the continuity of the partial derivatives to allow that $\lim_{t\to 0} \frac{\partial \mathrm{f}}{\partial \mathrm{x}}(x+\theta_1 t p_1, y+t p_2) = \frac{\partial \mathrm{f}}{\partial \mathrm{x}}(x, y)$ for example. We note the following restatement with $\mathbf{x} + t_0\mathbf{p}$ in place of $\mathbf{x}$:

$$\left(\frac{\mathrm{d}}{\mathrm{dt}}\right)_{t=t_0} f(\mathbf{x}+t\mathbf{p}) = \lim_{t\to t_0} \frac{\mathrm{f}(\mathbf{x} + \mathrm{t}\mathbf{p}) - \mathrm{f}(\mathbf{x} + \mathrm{t}_0\mathbf{p})}{\mathrm{t} - \mathrm{t}_0} = p_1 \frac{\partial \mathrm{f}}{\partial \mathrm{x}}(x+t_0 p_1, y+t_0 p_2)+p_2 \frac{\partial \mathrm{f}}{\partial \mathrm{y}}(x+t_0 p_1, y+t_0 p_2)$$

If now the continuity of the partial derivatives of f is assumed as it was to derive the directional derivative, then the Mean Value Proposition may be extended to the form of the Mean Value Theorem: For each $\mathbf{h}$ such that the line segment from $\mathbf{x}$ to $\mathbf{x} + \mathbf{h}$ lies in $\mathcal{O}$ there is a number $\theta \in (0, 1)$ such that

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = h_1 \frac{\partial \mathrm{f}}{\partial \mathrm{x}}(x + \theta h_1, y + \theta h_2) + h_2 \frac{\partial \mathrm{f}}{\partial \mathrm{y}}(x + \theta h_1, y + \theta h_2)$$

This follows by the previous restatement [(13.27) in text] upon defining $\phi(t) := f(\mathbf{x} + t\mathbf{p})$ and by applying the ordinary Lagrange Mean Value Theorem to $\phi(t)$, $0 \le t \le 1$.

Example. An application of the Mean value Theorem is that if the function f is continuously differentiable then the function is continuous. See Theorem 13.11.

Example. Find the direction in which $f(x, y) := x^2 + xy + 4y^2$ increases the fastest (with respect to changes in distance in the plane) at the point $(x, y) = 1, 2)$. What is this fastest rate of increase?
We have the gradient of f is $\nabla f(x, y) = (2x+y, x+8y)$. At the given point this becomes:$\nabla f(1, 2) = (4, 17)$. If $\mathbf{u}$ is a unit vector in the plane then the directional derivative of f in the direction of $\mathbf{u}$ at the given point is first equal to $\langle \nabla f(1, 2), \mathbf{u}\rangle = \langle (4, 17), (u_1, u_2)\rangle$ and second, this last expression is estimated by the Cauchy Schwarz inequality by $\langle (4, 17), (u_1, u_2)\rangle \le \sqrt{4^2 + 17^2} \cdot \sqrt{u_1^2 + u_2^2} = \sqrt{305} \cdot \sqrt{1}$. Thus the fastest rate of change is $\sqrt{305}$ and this is attained in the direction of $\mathbf{u} = (4/\sqrt{305}, 17/\sqrt{305})$ since equality is attained in the Cauchy-Schwarz inequality when the vectors are in the same direction.

Example. Problem 13.3 #11 (here for $n = 2$). Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable. Define $K := \{(x, y) : x^2 + y^2 \le 1\}$. (a) Since K is closed and bounded it is compact in $\mathbb{R}^2$ and therefore by By the Extreme Value Theorem there is a point $(x_0, y_0) \in K$ such that f attains its minimum over all $(x, y) \in K$ at (x_0, y_0). (b) Suppose in addition that if $\mathbf{u} \in \mathbb{R}^2$ is any unit vector then $\langle \nabla f(\mathbf{u}), \mathbf{u}\rangle > 0$. If $\|(x_0, y_0)\| = 1$ then we may apply this assumption to $\mathbf{u} = (x_0, y_0)$. Then since f is continuously differentiable everywhere we look at the directional derivative of f at (x_0, y_0) in the direction of $(-x_0, -y_0)$. We obtain $\frac{\mathrm{df}}{\mathrm{ds}} = \langle \nabla f(x_0, y_0), (-x_0, -y_0)\rangle = -\langle \nabla f(\mathbf{u}), \mathbf{u}\rangle < 0$. Therefore f decreases from the point (x_0, y_0) along a direction toward the origin. This is a contradiction to the definition of (x_0, y_0) as a minimizer. Therefore any minimizer must have norm strictly less than 1.

Example. 13.3 #7. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ has first order partial derivatives and satisfies

$$f(0, 0) = 1, \quad \frac{\partial \mathrm{f}}{\partial \mathrm{x}}(x, y) = 2, \quad \frac{\partial \mathrm{f}}{\partial \mathrm{y}}(x, y) = 3$$

Show that $f(x, y) = 1 + 2x + 3y$ for all $(x, y) \in \mathbb{R}^2$.
The result follows by the Mean Value Proposition 13.7. Indeed, apply this proposition with base point $\mathbf{x} = \mathbf{0}$ and an arbitrary displacement vector $\mathbf{h} = (h_1, h_2)$. Then $f(\mathbf{h}) - f(\mathbf{0}) = h_1 \frac{\partial \mathrm{f}}{\partial \mathrm{x}}(\theta_1 h_1, h_2) + h_2 \frac{\partial \mathrm{f}}{\partial \mathrm{y}}(0, \theta_2 h_2) = 2h_1 + 3h_2$. Since $f(0, 0) = 1$, we are done.