

1. Let X be a standard normal random variable (mean 0 and variance 1), and put $Y := X^2$. Show that the moment generating function of Y is

$$M_Y(t) = (1 - 2t)^{-1/2}.$$

Thus conclude that $E(X^4) = 3$.

2. Let U be a Chi-square random variable with 25 degrees of freedom. Use the Central Limit Theorem to approximate the following probability: $P(U > 35)$.

3. Four balls are placed in an urn. Each ball has a single number written on it. The numbers of the balls are respectively 1, 1, 2, and 4. Two balls are selected at random (without replacement) from the urn. Call the selected values X_1 and X_2 .

- (a) Find the probability mass function of the sample mean $\bar{X} := (X_1 + X_2)/2$.
(b) Calculate the mean and variance of \bar{X} directly from the probability mass function of part (a).
(c) Let μ and σ^2 denote the population mean variance respectively of the original population of four values. Show that the answers to part (b) satisfy $E(\bar{X}) = \mu$ and $Var(\bar{X}) = \sigma^2/3$. Would these answers be the same if the sampling had been done *with* replacement? Why or why not?

4. A certain housing development contains 8,000 condominium units. Suppose the actual fraction p of all condominium units that are planned to be sold within the coming year is unknown. A simple random sample of 400 of these units reveals that 44 of the 400 are planned to be sold.
- (a) Find a 90% confidence interval for p .
 - (b) Assume that $p \leq .20$. How large must the sample size be to estimate p within .02 with 95% confidence? First ignore the finite population correction in your calculation. Second, include this correction in a re-calculation of the sample size.

5. A population consists of $N = 5$ pairs (x, y) as follows: (1,1), (1,2), (2,4), (3,3), (3,5).
- (a) Show that the population parameters for means, variances and the covariance are as follows: $\mu_x = 2$, $\mu_y = 3$, $\sigma_x^2 = 4/5$, $\sigma_y^2 = 2$, and $\sigma_{xy} = 1$.
 - (b) . We take a simple random sample of the pairs of size 3. Show all 5 choose 3 (that is ten) possible random samples explicitly. Then compute the ratio estimator of μ_y , namely, $\bar{Y}_R := \mu_x \bar{Y} / \bar{X}$, for each of these ten possible samples.
 - (c) Find the actual bias and variance of \bar{Y}_R (these are different from the theoretical approximate bias and approximate variance). Then compare the MSE (mean squared error) of \bar{Y}_R with the MSE of \bar{Y} for estimating μ_y .

6. We are interested in the average value μ of an inventory of 1,000 items. We know that 75% of these items have a “low” value, while the other 25% have a “high” value, and thus we have two strata. We take a simple random sample of size n_L from the stratum of low values, and independently a sample of size n_H from the stratum of high values. Assume that the total sample size is $n = n_L + n_H = 100$. Assume that the subpopulation standard deviations of the inventory values are $\sigma_L = \sigma$ and $\sigma_H = 2\sigma$ for some $\sigma > 0$. Define the stratified estimator of μ by

$$\bar{X}_s := .75\bar{X}_L + .25\bar{X}_H$$

where \bar{X}_l is the sample mean of values for sample size n_l , $l = L, H$. Calculate the mean and variance of \bar{X}_s in terms of σ in each of the following cases.

(a) $n_L = n_H = 50$.

(b) $n_L = 75$ and $n_H = 25$.

(c) Find the optimal allocation for a total sample size of $n = 100$, and then compute the variance of \bar{X}_s in this case as well.