Math 481/581 Mathematical Statistics I.

Notes 3

(p.1)

Generating Samples from a Given Distribution

Suppose the distribution function $G(x)$ of a random variable is given. How could we simulate random values for this distribution? We will assume that pseudo-random numbers can be generated that will look like an independent random sample from the uniform distribution on the unit interval (cf: Chap 2 Exercise 72). How can we suitably transform these pseudo-random values to obtain a nearly independent random sample with cumulative distribution function G?

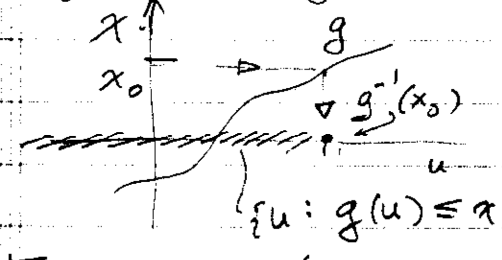
The Distribution Function Method

Suppose that $U \sim \text{Uniform}\,[0,1]$. Put $X = 25\,U + 75$. What is the distribution of X?

Fix $x_0 \in \mathbb{R}$. We write $F(x_0) = P(X \le x_0) = P(25\,U + 75 \le x_0)$

Now $x = g(u) = 25u + 75$ is an increasing function of u. Therefore $\{u : 25u + 75 \le x_0\}$

$= \{u : u \le g^{-1}(x_0)\} = \{u : u \le \frac{x_0 - 75}{25}\}$.



$\{u : g(u) \le x_0\} = \{u : u \le g^{-1}(x_0)\}$.

$\therefore\; F(x_0) = P\left(U \le \frac{x_0 - 75}{25}\right) = \frac{x_0 - 75}{25}$ as long as $\frac{x_0 - 75}{25} \in [0,1]$

by uniform distribution

(over)

Therefore we have $F(x) = \frac{x-75}{25}$, $75 \le x \le 100$.
So the density of X is

$$f(x) = \frac{d}{dx} F(x) = \begin{cases} \frac{1}{25} & 75 \le x \le 100 \\ 0 & \text{else} \end{cases}$$

So $X \sim$ Uniform$[75, 100]$.

Thus if we want to simulate random values uniformly distributed in the interval $[75, 100]$ we simply transform pseudo-random numbers u in $[0,1]$ by $g(u) = 25u + 75$. Notice that the argument above yields the following. If $g(u)$ is strictly increasing then the distribution function of $X := g(U)$ is simply $F(x) = g^{-1}(x)$ for all x such that $u = g^{-1}(x) \in [0,1]$. In particular if we take $g(u) = G^{-1}(u)$, for a given distribution function G, then $g^{-1}(x) = (G^{-1})^{-1}(x) = G(x)$
So

$$X := G^{-1}(U) \text{ has cdf } G(x).$$

(This is proposition D, p.63)
To explain this result, note that $G^{-1}(u)$ is the u^{th} quantile of the distribution G, for $u \in [0,1]$. But u is the u^{th} quantile of the uniform distribution on $[0,1]$. Therefore we are simply mapping $u =$ the u^{th} quantile of one distribution (the uniform one) to $x = G^{-1}(u) =$ the u^{th} quantile of another distribution (the distribution G)
Thus $X = G^{-1}(U)$ is sometimes called the Quantile transform.

Notes 3.

(p.3)

2.3 Example E. Simulate an exponential r.v. with parameter λ as follows: The distribution function of an exponential r.v. is:

$$F(x) = 1 - e^{-\lambda x}, \quad x \geq 0. \text{ (see p. 50)}$$

The quantile transform is the inverse of the relation

$$u = 1 - e^{-\lambda x} \iff e^{-\lambda x} = 1 - u$$

$$\iff -\lambda x = \ln(1-u) \iff x = -\frac{1}{\lambda}\ln(1-u).$$

Therefore $X := -\frac{1}{\lambda}\ln(1-U)$ is distributed as an exponential r.v. with parameter λ, where $U \sim$ Uniform $[0,1]$.

Further Examples of the Distribution function Method.

Let $$\Phi(z) = \int_{-\infty}^{z} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\, dt$$

be the standard normal cumulative distribution function (tabulated for $z \geq 0$ in Appendix B, p. A7)

Suppose $X \sim N(\mu, \sigma^2)$. Then the cumulative distribution function of X is: $F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(t-\mu)^2/2\sigma^2}\, dt.$

Show that $Z := \frac{X-\mu}{\sigma}$ has cdf. Φ.

We have $P(Z \leq z) = P\left(\frac{X-\mu}{\sigma} \leq z\right) = P(X \leq \sigma z + \mu)$

$= F(\sigma z + \mu)$. Therefore the density of Z is:

$$\varphi(z) = \frac{d}{dz}F(\sigma z+\mu) \underset{\text{chain rule}}{=} F'(\sigma z+\mu)\cdot\sigma = \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} e^{-(\sigma z+\cancel{\mu}-\cancel{\mu})^2/2\sigma^2}\cdot\cancel{\sigma}$$

$= \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$. Thus, because $\varphi(z) = \Phi'(z)$ we are done.