Notes 2

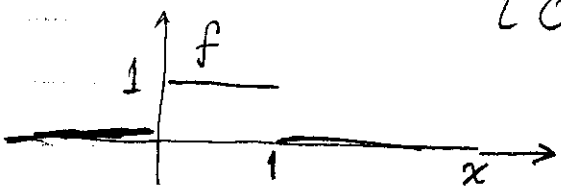
Math 481/581 Mathematical Statistics I

p. 1

Continuous random variable.

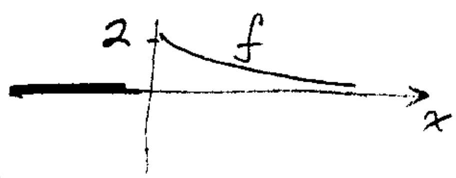
A random variable X is said to be continuous if the cumulative distribution function $F(x) = P(X \le x)$ is continuous as a function of x. However for the purposes of most applications a special type of continuous r.v. is considered only. We assume that there exists a density function $f(x)$ that is non-negative ($f \ge 0$) and piecewise continuous except for jump discontinuities with $\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$ and such that $\boxed{F(x) = \int_{-\infty}^{x} f(t)\,dt.}$

Examples. (a) $f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{all other } x \end{cases}$



$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 1\,dx = 1$$

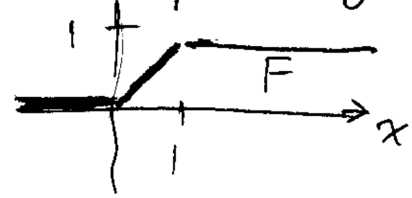
(b) $f(x) = \begin{cases} 2e^{-2x} & x \ge 0 \\ 0 & \text{all other } x \end{cases}$



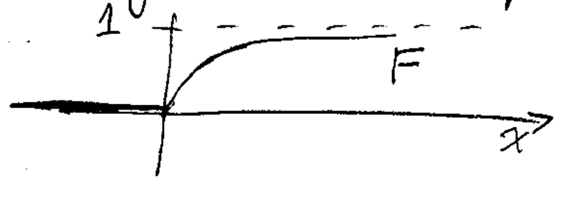
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} 2e^{-2x}\,dx \underset{=}{\boxed{\begin{array}{l} u=2x \\ du=2dx \end{array}}} \int_0^{\infty} e^{-u}\,du = -e^{-u}\Big]_0^{\infty}$$
$$= \left[e^{-u}\right]_{\infty}^{0} = e^{-0} - e^{-\infty} = 1 - 0 = 1.$$

This density plays the role of the derivative of $F(x)$ wherever $F'(x)$ exists: $F'(x) = f(x)$ for all points of continuity of $f(x)$.

Graph of $F(x)$ in examples above: (a)



(b)



(p.2)

Since we have $P(a < X \le b) = F(b) - F(a)$

$= \int_a^b f(x)dx$ then we also have

$$\boxed{P(a < X < b) = \int_a^b f(x)\,dx}$$ since $P(X = b) =$
$F(b) - F(b-) = F(b) - F(b) = 0$ by continuity of $F(x)$. In other words a continuous r.v. has zero probability to be equal to any single given value x: $P(X = x) = 0 \quad \forall x$.

Chapter 2 Exercise 39. The Cauchy cumulative distribution function is $F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x) \quad -\infty < x < \infty$

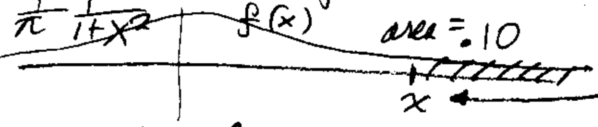
(a) Obviously $F(x)$ is continuous and non-decreasing. We verify further that $\lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x)$

$= \frac{1}{2} + \frac{1}{\pi}\left(-\frac{\pi}{2}\right) = \frac{1}{2} - \frac{1}{2} = 0$, and

$\lim_{x \to +\infty} F(x) = \lim_{x \to +\infty} \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x) = \frac{1}{2} + \frac{1}{\pi}\frac{\pi}{2} = \frac{1}{2} + \frac{1}{2} = 1.$

(b) Since $F(x)$ is everywhere differentiable we have that the density exists and is given by $f(x) = F'(x) = \frac{1}{\pi}\frac{1}{1+x^2}$, $-\infty < x < \infty$.

(c) We want to find x so that $P(X > x) = 0.1$



This value of x is called the 90^{th} percentile, 0.9 quantile

We must solve $1 - P(X \le x) = 0.1$ or
$P(X \le x) = 0.9$. That is $F(x) = 0.9 \iff$
$\frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x) = 0.9 \iff \tan^{-1}(x) = (0.4)\pi$
$\iff x = \tan((0.4)\pi) = 3.078$. We can write

F-inverse $\rightarrow$ $\boxed{F^{-1}(0.9) = 3.078}$

(p.3)

<u>Interpretation of the density</u> If f is continuous at x then the probability that the r.v. X is near to x may be written
$$P\left(x-\frac{\delta}{2} \le X \le x+\delta/2\right) = \int_{x-\frac{\delta}{2}}^{x+\delta/2} f(u)\,du$$
$\approx \delta f(x)$. Alternatively, in infinitesimal notation,

(*) $\boxed{P(x \le X \le x+dx) \approx f(x)\,dx.}$

So $f(x)$ is the instantaneous rate at which probability accrues. It has units of inverse units of x. The formula (*) shows that $f(x)dx$, a probability, (whereas $f(x)$ itself is <u>not</u> a probability) plays the role of the frequency function quantity $p(k)$ that is also a probability. So in our formula for calculating probabilities it is natural that summation of the $p(k)$ in the discrete case is replaced by continuous summation of the $f(x)dx$ in the continuous case.

<u>Example</u>. (i) Show that
$$f(x) = \begin{cases} \alpha x^{-\alpha-1}, & x \ge 1 \\ 0 & \text{else} \end{cases}$$
is a probability density for any $\alpha > 0$ and,
(ii) find the cumulative distribution function $F(x)$.

(i) obviously $f(x) \ge 0$. We have $\int_{-\infty}^{\infty} f(x)\,dx$
$$= \int_1^\infty \alpha x^{-1-\alpha}\,dx = \frac{\alpha}{-\alpha} x^{-\alpha}\Big]_1^\infty = x^{-\alpha}\Big]_\infty^1$$
$= 1^{-\alpha} - \infty^{-\alpha} = 1-0 = 1$, for any fixed $\alpha > 0$.

(ii) $$F(x) = \int_{-\infty}^{x} f(u)\,du = \begin{cases} 0 & x < 1 \\ \int_1^x \alpha u^{-1-\alpha}\,du, & x \ge 1 \end{cases}$$

where $\int_1^x \alpha u^{-1-\alpha}\,du = -u^{-\alpha}\Big]_1^x = u^{-\alpha}\Big]_x^1 = \boxed{1 - x^{-\alpha},\ x \ge 1}$