Math 481/581 Mathematical Statistics I

Notes 13

(p. 1) A random sum of a sequence of independent identically distributed random variables.

Let $X_1, X_2, X_3, \ldots$ be independent r.v.'s each distributed as an exponential variable with parameter λ, and let N be an independent Poisson r.v. with parameter μ. Define

$$S = \sum_{i=1}^{N} X_i \quad \text{(a random sum)}$$

Compute the moment generating function of S as follows. $M_S(t) = E(e^{tS}) =$

$$E\left(E\left(e^{tS} \mid N\right)\right) = E\left(E\left(e^{t(X_1+\cdots+X_N)} \mid N\right)\right)$$

$$= E\left\{\left(\frac{\lambda}{\lambda-t}\right)^N\right\} = E\left(e^{N\ln\left(\frac{\lambda}{\lambda-t}\right)}\right)$$

by Property D

$$= M_N\left(\ln\left(\frac{\lambda}{\lambda-t}\right)\right)$$

Here $M_N(s)$ is given in Example A, p. 156 by $M_N(s) = e^{\mu(e^s-1)}$.

So, $M_S(t) = e^{\mu\left(e^{\ln\left(\frac{\lambda}{\lambda-t}\right)}-1\right)} = e^{\mu\left(\frac{\lambda}{\lambda-t}-1\right)}$

$$= e^{\mu\frac{t}{\lambda-t}}$$

We compute $M_S'(t) = \mu \cdot \frac{\lambda}{(\lambda-t)^2} e^{\mu\left(\frac{t}{\lambda-t}\right)}$

so $E(S) = M_S'(0) = \mu \cdot \frac{\lambda}{\lambda^2} = \mu/\lambda \quad (= E(X_1)E(N))$

$M_S''(t) = \left[\mu^2 \cdot \frac{\lambda^2}{(\lambda-t)^4} + 2\mu\frac{\lambda}{(\lambda-t)^3}\right] e^{\mu\left(\frac{t}{\lambda-t}\right)}$. So,

$M_S''(0) = \frac{\mu^2}{\lambda^2} + \frac{2\mu}{\lambda^2}$. $\therefore$ $\boxed{Var(S) = \frac{2\mu}{\lambda^2} = E(X_1)^2 Var(N) + E(N) Var(X_1)}$

Note that $E(N) = Var(N) = \mu$ and $E(X_1) = 1/\lambda$, $Var(X_1) = 1/\lambda^2$ (Example E, pp. 150-151)

Notes

(p2) Approximate mean and Variance of a function of a r.v.

Suppose $E(X)=\mu$ and $Var(X)=\sigma^2$. What can be said about the mean and variance of $Y=g(X)$?

Example. Let $X \sim$ Poisson(λ) and Put $Y=\sqrt{X}$. We have $\mu=\mu_X=\lambda$ and $\sigma_X^2=\lambda$. What about Y? Write out a Taylor series for $y=g(x)$ about $x=\mu$:

$$(*)\quad \sqrt{x} \approx \sqrt{\mu} + \underbrace{\tfrac{1}{2}\mu^{-1/2}}_{g'(\mu)}(x-\mu) + \frac{1}{2!}\underbrace{\left(-\tfrac{1}{4}\mu^{-3/2}\right)}_{g''(\mu)}(x-\mu)^2 + \cdots$$

Therefore

$$E(\sqrt{X}) \approx \sqrt{\mu} + 0 - \frac{1}{8}\mu^{-3/2}\, Var(X)$$

$$= \sqrt{\mu} - \frac{1}{8}\mu^{-3/2}\cdot\mu = \sqrt{\mu} - \frac{1}{8\sqrt{\mu}}.$$

Also, using just the first two terms of $(*)$, argue that $Var(\sqrt{X}) \approx \left(\frac{1}{2}\mu^{-1/2}\right)^2 Var(X)$

$$= \frac{1}{4}\mu^{-1}\cdot\mu = \frac{1}{4}.$$

Thus $Y=\sqrt{X}$ has nearly constant variance (independent of μ)

In summary we have: $\boxed{E(Y) \approx g(\mu_X) + \frac{1}{2}g''(\mu_X)\,\sigma_X^2 \text{ and } Var(Y) \approx g'(\mu_X)^2\cdot\sigma_X^2.}$

(6.3) Example. Let $Y = X^4$ where $X \sim \text{Gamma}(\lambda, \alpha)$.

Then $E(X) = \frac{\alpha}{\lambda}$ and $Var(X) = \frac{\alpha}{\lambda^2}$.

Therefore
$$E(Y) \approx \left(\frac{\alpha}{\lambda}\right)^4 + \frac{1}{2} 12 \left(\frac{\alpha}{\lambda}\right)^2 \left(\frac{\alpha}{\lambda^2}\right) = \frac{\alpha^4 + 6\alpha^3}{\lambda^4}.$$

and
$$Var(Y) \approx \left[4\left(\frac{\alpha}{\lambda}\right)^3\right]^2 \cdot \frac{\alpha}{\lambda^2} = 16 \frac{\alpha^7}{\lambda^8}$$

The exact values are as follows

$$E(Y) = \int x^4 \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx$$
$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(4+\alpha)}{\lambda^{\alpha+4}} = \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{\lambda^4}$$
$$= \left(\frac{\alpha}{\lambda}\right)^4 + 6\left(\frac{\alpha^3}{\lambda^4}\right) + \frac{11\alpha^2}{\lambda^4} + \frac{6\alpha}{\lambda^4}.$$

and
$$E(Y^2) = \int x^8 \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx$$
$$= \frac{\alpha(\alpha+1)\cdots(\alpha+7)}{\lambda^8} \quad \text{so}$$

$$Var(Y) = \frac{\alpha(\alpha+1)\cdots(\alpha+7) - [\alpha(\alpha+1)(\alpha+2)(\alpha+3)]^2}{\lambda^8}$$
$$= \frac{(\alpha^8 + 28\alpha^7 + \cdots) - (\alpha^8 + 12\alpha^7 + \cdots)}{\lambda^8}$$
$$= \frac{16\alpha^7}{\lambda^8} + \cdots$$

Therefore the approximations are good to leading order in α.

(4)

Functions of two random variables

Let X and Y be given with respective means, standard deviations and covariance given by
$\mu_1, \mu_2, \sigma_1, \sigma_2$ and σ_{12} $(= \rho\sigma_1\sigma_2)$

How can we approximate the mean and variance of $Z_1 = g(X, Y)$ for some real valued function g of two variables.

Example. Let X and Y be independent r.v.'s with $X = 1 + \text{Poisson}(\lambda)$ and $Y = 1 + \text{Poisson}(\lambda)$.

Then $Z = Y/X$ has finite moments of all orders.

Use Taylor's expansion $g(x,y) = g(\mu_1,\mu_2)$

$$+\frac{\partial g}{\partial x}(\mu_1,\mu_2)(x-\mu_1) + \frac{\partial g}{\partial y}(\mu_1,\mu_2)(y-\mu_2)$$

$$+ \frac{1}{2}\frac{\partial^2}{\partial x^2}g(\mu_1,\mu_2)(x-\mu_1)^2 + \frac{\partial^2}{\partial x \partial y}g(\mu_1,\mu_2)(x-\mu_1)(y-\mu_2)$$

$$+ \frac{1}{2}\frac{\partial^2}{\partial y^2}g(\mu_1,\mu_2)(y-\mu_2)^2 + \cdots$$

$\frac{\partial g}{\partial x} = \frac{-y}{x^2}$, $\frac{\partial g}{\partial y} = \frac{1}{x}$

$\frac{\partial^2 g}{\partial x \partial y} = -\frac{1}{x^2}$

$\frac{\partial^2 g}{\partial x^2} = \frac{2y}{x^3}$, $\frac{\partial^2 g}{\partial y^2} = 0$

It follows that $E(Z_1) \approx \frac{\mu_2}{\mu_1} + 0 +$ (since $E(X-\mu_1) = E(Y-\mu_2) = 0$)

$$\frac{1}{2}\left(\frac{2\mu_2}{\mu_1^3}\sigma_1^2 + 2\left(-\frac{1}{\mu_1^2}\right)\sigma_{12} + 0\cdot\sigma_2^2\right) = \frac{\mu_2}{\mu_1} + \frac{\mu_2}{\mu_1^3}\sigma_1^2$$

(since here $\sigma_{12} = 0$)

$$= \frac{1+\lambda}{1+\lambda} + \frac{1+\lambda}{(1+\lambda)^3}\cdot\lambda = \boxed{1 + \frac{\lambda}{(1+\lambda)^3}}$$

and $Var(Z_1) \approx \left(-\frac{\mu_2}{\mu_1^2}\right)^2 \sigma_1^2 + \left(\frac{1}{\mu_1}\right)^2\sigma_2^2$

$$+ 2\left(\frac{-\mu_2}{\mu_1^2}\right)\left(\frac{1}{\mu_1}\right)\sigma_{12} = \left(\frac{1}{1+\lambda}\right)^2\left(1+\lambda + 1+\lambda - 2\cdot 0\right) = \boxed{\frac{2}{1+\lambda}}.$$