Math 481/581 Mathematical Statistics I

Notes 12

(p.1)

The Moment generating function

Given a random variable X, define the so-called moment generating function of X by

$$M(t) = E\left(e^{tX}\right)$$

$$= \begin{cases} \sum_x e^{tx}\, p(x) & \text{(discrete case)} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx & \text{(continuous case)} \end{cases}$$

for all t such that the sums exist.

Example. Let $X \sim$ binomial $(2, \frac{1}{2})$,

so that the frequency function is:

x	0	1	2
$p(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$$M(t) = \sum_{x=0}^{2} e^{tx} p(x) = p(0)e^{t\cdot 0} + p(1)e^{t\cdot 1} + p(2)e^{t\cdot 2}$$

$$= \frac{1}{4}\cdot 1 + \frac{1}{2}\, e^t + \frac{1}{4} e^{2t}$$

Note that $M'(t) = 0 + \frac{1}{2}\cdot 1\, e^t + \frac{1}{4}\cdot 2\, e^{2t}$

and $M''(t) = 0 + \frac{1}{2}\cdot 1\cdot 1\, e^t + \frac{1}{4}\cdot 2\cdot 2\cdot e^{2t}$

$\therefore$ $M'(0) = 0 + \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 = E(X) = 1$

$M''(0) = 0 + \frac{1}{2}\cdot 1^2 + \frac{1}{4}\cdot 2^2 = E(X^2) = 3/2$

$Var(X) = M''(0) - (M'(0))^2 = \frac{3}{2} - 1^2 = \frac{1}{2}$

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It follows in general that $M^{(r)}(0) = E(X^r)$, $r = 1, 2, \ldots$
(as long as $M(t)$ exists in an open interval about $t=0$)

We generalize the above example as follows.

Let $X \sim$ binomial (n, p). Then

$$M(t) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} e^{tx}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x} = (pe^t + (1-p))^n$$

Binomial Theorem.

$$M'(t) = n(pe^t + (1-p))^{n-1} \cdot pe^t$$

$$M''(t) = n(pe^t + (1-p))^{n-1} \cdot pe^t + n(n-1)(pe^t + (1-p))^{n-2} \cdot (pe^t)^2$$

$$\therefore M'(0) = n(1)^{n-1} \cdot p = \boxed{np = E(X)}$$

$$M''(0) = n\, 1^{n-1} \cdot p + n(n-1)\, 1^{n-2} \cdot p^2$$

$$= np + (n^2 - n)p^2.$$

Hence $Var(X) = M''(0) - M'(0)^2 = np + (n^2 - n)p^2 - n^2p^2$

$$= np - np^2 = n(p - p^2) = \boxed{np(1-p)}.$$

Example. Let $X \sim Gamma(\lambda, \alpha)$.

Then $M(t) = \int_0^\infty e^{tx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx$

$M(t)$ exists for all $t < \lambda$ since then we still have a negative exponential term in the integrand.

Now we use the identity: $\int_0^\infty x^{\alpha-1} e^{-\lambda_1 x} dx = \frac{\Gamma(\alpha)}{\lambda_1^\alpha}$

with $\lambda_1 = (\lambda - t)$ to obtain:

$$M(t) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \Gamma(\alpha) / (\lambda - t)^\alpha = \left(\frac{\lambda}{\lambda - t}\right)^\alpha, \quad (t < \lambda).$$

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In particular when $\alpha = 1$ we have the exponential variable with parameter λ whose moment generating function is thereby $M(t) = \frac{\lambda}{\lambda - t}$, $t < \lambda$.

Now suppose X and Y are independent random variables. Let $Z := X + Y$. Then (Property D, p.159):

$$M_Z(t) = E(e^{tZ}) = \int_{-\infty}^{\infty} e^{tz} f_Z(z)\,dz$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t(x+y)} f(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{tx}e^{ty} f_X(x) f_Y(y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx \int_{-\infty}^{\infty} e^{ty} f_Y(y)\,dy = M_X(t) M_Y(t).$$

Also Property A, p.155 states that the moment generating function uniquely determines the distribution as long as $M(t)$ exists in an open interval about $t=0$.

Therefore we note the following application of Properties A and D. Let $\alpha = n$, a positive integer. Then, since

$$\left(\frac{\lambda}{\lambda - t}\right)^{\alpha} = \left(\frac{\lambda}{\lambda - t}\right)^{n} = \left[M_{Exp(\lambda)}(t)\right]^n,$$

we have that a Gamma(λ, n) distribution is the distribution of the sum of n independent exponential r.v.'s each with parameter λ.

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p.41 **Moment generating function of $N(\mu, \sigma^2)$**

Let $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty$

Then $$M(t) = \int_{-\infty}^{\infty} \frac{e^{tx}}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2} dx$$

Put $z = \frac{x-\mu}{\sigma}$, $dz = \frac{dx}{\sigma}$, $x = \sigma z + \mu$ $\quad z: -\infty \to \infty$

$$M(t) = \int_{-\infty}^{\infty} e^{t(\sigma z+\mu)} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz = e^{\mu t} M_Z(\sigma t)$$

where $M_Z(s) = \int_{-\infty}^{\infty} e^{sz} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz$ is the m.g.f. of $N(0,1)$.

Now $$M_Z(s) = \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(z^2-2sz)}}{\sqrt{2\pi}} dz = e^{\frac{1}{2}s^2} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(z^2-2sz+s^2)}}{\sqrt{2\pi}} dz$$

$$= e^{\frac{1}{2}s^2} \int_{-\infty}^{\infty} \frac{e^{-(z-s)^2/2}}{\sqrt{2\pi}} dz = e^{\frac{1}{2}s^2} \cdot 1 = e^{\frac{1}{2}s^2}.$$

Therefore $$M_{N(\mu,\sigma^2)}(t) = e^{\mu t} e^{\frac{1}{2}(\sigma t)^2} = \boxed{e^{\mu t + \frac{1}{2}\sigma^2 t^2}}$$

Note that indeed $M'(t) = (\mu + \sigma^2 t) e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

and $M''(t) = \left[(\mu+\sigma^2 t)^2 + \sigma^2\right] e^{\mu t + \frac{1}{2}\sigma^2 t^2}$;

so $E(X) = \mu e^0 = \mu$ and $E(X^2) = \mu^2 + \sigma^2$

Therefore $Var(X) = \mu^2 + \sigma^2 - (\mu)^2 = \sigma^2$. Hence μ and σ^2 actually are the mean and variance of the $N(\mu, \sigma^2)$ distribution.