(1) Math 481/581 Math Stat I Fall 2007

~~Chapter 2 Examples~~

Chap. 2 #13 (a) $X \sim$ binomial $(n=20, p=1/3)$

$$P(X \geq 12) = \sum_{x=12}^{20} P(X=x) = \sum_{x=12}^{20} \binom{20}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{20-x}$$

$$= \binom{20}{12}\left(\frac{1}{3}\right)^{12}\left(\frac{2}{3}\right)^{8} + \binom{20}{13}\left(\frac{1}{3}\right)^{13}\left(\frac{2}{3}\right)^{7} + \binom{20}{14}\left(\frac{1}{3}\right)^{14}\left(\frac{2}{3}\right)^{6} + \cdots$$

$$= \frac{(125970)\cdot 2^8}{3^{20}} + \frac{77520}{3^{20}}\cdot 2^7 + \frac{38760}{3^{20}}\cdot 2^6 + \frac{15504}{3^{20}}\cdot 2^5 + \ldots$$

$$= .009248 + .002845 + .00071 + .00014 + \ldots$$

$$\approx .0130$$

Chap 2 #13 (b) $p = 1/2$ in place of $p = 1/3$

$$P(X \geq 12) = \frac{125970}{2^{20}} + \frac{77520}{2^{20}} + \cdots = .12013$$

$$+ .07393 + .03696 + .01479 + .00462 + \ldots \approx .252$$

Example A pair of fair dice is thrown repeatedly until the sum of the two dice equals 7.
What is the probability that the pair of dice is thrown
(a) exactly 4 times
(b) 5 times or more.

$p = \text{Prob(sum of dice} = 7) = \frac{6}{36} = 1/6.$

X = # of times pair is thrown $\sim$ geometric $(p = 1/6)$

(a) $P(X = 4) = (1-p)^3 \cdot p = (5/6)^3 \cdot 1/6 = .096$ (over)

Chapter 2 examples (p.2)

(b) $P(X \geq 5) = P(\overbrace{F_1 F_2 F_3 F_4}^{\text{fail to throw a "7" on each of first 4 trials}})$

$= (1-p)^4 = (5/6)^4 = .482.$

Chap 2 #25 $p = \text{Prob.(royal flush)} = \dfrac{4}{\binom{52}{5}} = 1.54 * 10^{-6}.$

(Note correction to text).

(a) $n = 20 * 52 * 100 = 104{,}000$ trials

X = # of time royal flush appear

$\sim$ binomial $(10400,\ 1.54 * 10^{-6})$

$P(X=0) = \binom{n}{0} p^0 (1-p)^n = (.99999846)^{104000}$

$= .852$

Poisson approximation $\lambda = np = 0.16$

$P(X=0) \approx e^{-\lambda} \dfrac{\lambda^0}{0!} = e^{-0.16} = .852.$

(b) $P(X=2) \approx e^{-\lambda} \dfrac{\lambda^2}{2!} = e^{-0.16} \dfrac{(0.16)^2}{2!} = .011.$

Example (Compare Chap. 2 #27). Suppose a rare disease has a rate of incidence 1 in 100,000. / per year

Find the probability of k cases this year in a population of size 200,000.

$X \sim$ binomial $(200{,}000,\ 10^{-5}) \ \underset{\text{approx.}}{\dot{\sim}}\ \text{Poisson}(\lambda = 2)$

$P(X=k) \approx e^{-2} \dfrac{2^k}{k!}$.

k	0	1	2
$P(X=k)$	$e^{-2} = .1353$	$2e^{-2} = .2706$	$\frac{2^2}{2} e^{-2} = .2706$

Chap 2 #31 Rate of calls $\lambda = 2$ per hours.
In a ten minute period $\lambda_1 = \left(\frac{1}{6}\text{ hr}\right)\cdot\lambda$
$= 1/3$ (calls)

(a) $X \sim \text{Poisson}(\lambda_1)$ $\quad P(X \geq 1)$
$= 1 - P(X=0) = 1 - e^{-\lambda_1} = 1 - e^{-1/3} = .2834$

(b) Let t hours = length of shower
$\lambda_1 = t\cdot\lambda = 2t$ = expected number of calls.
$X \sim \text{Poisson}(\lambda_1)$ is number of calls
Want $P(X=0) \geq .5$ or $e^{-2t} \geq .5$
$\ln(e^{-2t}) \geq \ln(.5)$
$\Leftrightarrow -2t \geq -\ln 2 \Leftrightarrow 2t \leq \ln 2$
$\Leftrightarrow t \leq \frac{\ln 2}{2} = .3466$ hr. $= 20.8$ min.

Chap 2. #6 Let T be exponential density $f(t) = \lambda e^{-\lambda t}, t \geq 0$.
Assume $P(T<1) = .05$. Find λ.

$P(T<1) = .05 = \int_0^1 \lambda e^{-\lambda t}\,dt = \int_0^1 d(-e^{-\lambda t})$
$= \left[-e^{-\lambda t}\right]_0^1 = \left.e^{-\lambda t}\right]_1^0 = e^{-\lambda\cdot 0} - e^{-\lambda\cdot 1} = 1 - e^{-\lambda}$

$\therefore .05 = 1 - e^{-\lambda} \Leftrightarrow e^{-\lambda} = .95$
$\Leftrightarrow -\lambda = \ln(.95) = -.0513$
So $\boxed{\lambda = .0513}$

Chap 2 Example (4)

Example Show that $\int_0^\infty \frac{\lambda^3 t^2 e^{-\lambda t}}{2!}\,dt = 1$.
(compare Chap 2 #48)

Gamma density ($\alpha = 3$, λ)

By a change of variables $u = \lambda t$ we find that $\boxed{\lambda^2 t^2 = u^2}$ and $\boxed{du = \lambda\, dt}$, so integral becomes $\int_0^\infty \frac{u^2 e^{-u}}{2!}\,du$

Now $\int_0^\infty u^2 e^{-u}\,du = \int_0^\infty u^2\, d(-e^{-u}) = -u^2 e^{-u}\Big]_0^\infty$

$+ \int_0^\infty 2u\, e^{-u}\,du = \int_0^\infty 2u\, e^{-u}\,du$

$= 2\int_0^\infty u\, d(-e^{-u}) = 2\left[-e^{-u} u\right]_0^\infty + 2\int_0^\infty e^{-u}\,du$

$= 0 + 2\cdot 1 = 2!$ Therefore we have shown that indeed the Gamma density with shape parameter $\alpha = 3$ and scaling parameter λ integrates to 1.

Chap 2 #55 Let $X \sim N(\mu, \sigma^2)$. Find c so that $P(\mu - c \le X \le \mu + c) = 0.95$

The probability may be written

$$P\left(\frac{\mu - c - \mu}{\sigma} \le \frac{X - \mu}{\sigma} \le \frac{\mu + c}{\sigma}\right)$$

$$= P\left(-\frac{c}{\sigma} \le N(0,1) \le \frac{c}{\sigma}\right) = \Phi\left(\frac{c}{\sigma}\right) - \Phi\left(-\frac{c}{\sigma}\right)$$

$$= \Phi\left(\frac{c}{\sigma}\right) - \left(1 - \Phi\left(\frac{c}{\sigma}\right)\right) = 2\Phi\left(\frac{c}{\sigma}\right) - 1 = 0.95 \Leftrightarrow \frac{c}{\sigma} = \Phi^{-1}(.975) = 1.96.$$