Math 481/581 Mathematical Statistics I

Notes 5

(p1)

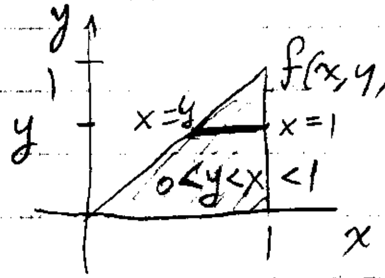
The method of conditional density.

Suppose X and Y are jointly distributed as follows. X is uniformly distributed on $(0,1)$, and given $X=x$, Y is uniformly distributed on $(0,x)$. Thus we are given

$$f_X(x) = 1, \quad 0<x<1, \text{ and } f_{Y|X}(y|x) = \frac{1}{x}, \; 0<y<x.$$

Thus the joint density is the product:

$$f(x,y) = f_{Y|X}(y|x)\, f_X(x) = \begin{cases} 1\cdot\frac{1}{x} & 0<y<x<1 \\ 0 & \text{else.} \end{cases}$$



$f(x,y) > 0$ on a triangle as shown.

We find the marginal density of Y as:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_y^1 \frac{1}{x}\,dx = \ln x \Big]_y^1$$

$$= \ln 1 - \ln y = -\ln(y), \quad 0<y<1.$$

Hence the conditional density of X given $Y=y$ is:

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} \dfrac{1}{-x\ln(y)} & , y<x<1. \\ 0 & \text{all other } x \end{cases}$$

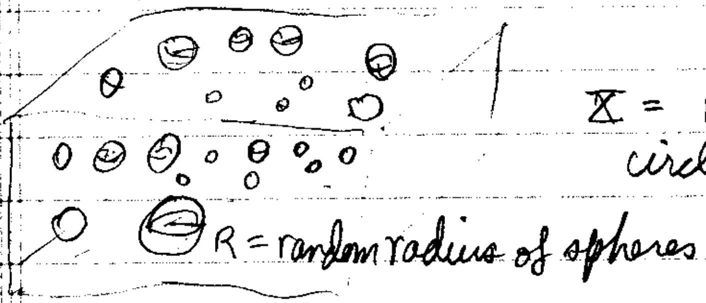
Hence $E(X|Y=y) = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\,dx$ integrand is constant in x!

$$= \int_y^1 x\left(-\frac{1}{x\ln y}\right)dx = \int_y^1 -\frac{1}{\ln(y)}\,dx = \frac{1-y}{-\ln(y)}$$

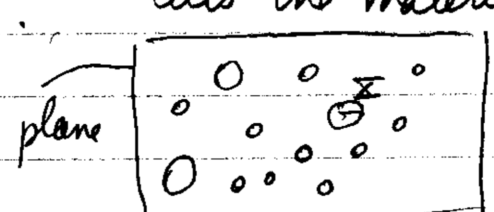
at $y=.44$ this becomes $\frac{.56}{-\ln(.44)} = \boxed{.6821}$

(p. 2)

3.5.2 Example B.



X = random radius of cross-sectional circles when a random plane cuts the material.



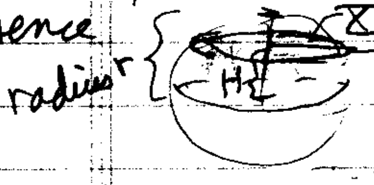
The holes in a slice of Swiss cheese.

Consider a subset of the spheres defined by the condition:

$R \in [r, r+dr]$,

that is R is infinitesimally close to r for some fixed radius r. So all the spheres that remain in the material with this condition have the same radii. The plane cutting at random is the condition that the height H at which these spheres is cut is uniformly distributed on $[0, r]$.

~~Hence~~



cross section of radius X given $R=r$

Since $f_H(h) = \frac{1}{r} \quad 0 < h < r$

and since given $R=r$, $X^2 + H^2 = r^2$, we have the conditional cdf of X given $R=r$ is

$$F_{X|R}(x|r) = P(X \le x \mid R=r)$$

$$= P(\sqrt{r^2-H^2} \le x)$$

$$= P(H \ge \sqrt{r^2-x^2}) = \int_{\sqrt{r^2-x^2}}^{r} \frac{1}{r}\, dh$$

constant integrand!

$= 1 - \frac{\sqrt{r^2-x^2}}{r}$. Thus the conditional density of X given $R=r$ is $f_{X|R}(x|r) = \frac{d}{dx}\left(1 - \frac{\sqrt{r^2-x^2}}{r}\right) = \frac{x}{r\sqrt{r^2-x^2}} \quad 0 < x < r$

Thus $f(x,r) = \frac{x}{r\sqrt{r^2-x^2}} f_R(r)$ and $f_X(x) = \int_x^\infty \frac{x}{r\sqrt{r^2-x^2}} f_R(r)\, dr$, See p. 91.

(p.3)

<u>Bayesian Inference</u>. If θ is a parameter for a density $f(x|\theta)$, then we may regard θ as an unknown and so give θ an initial density $g(\theta)$. Then the joint density of X and Θ is $f(x,\theta) = f(x|\theta)g(\theta)$

so the marginal density of X is

$f_X(x) = \int f(x|\theta)g(\theta)\,d\theta$ and the so-called posterior density of Θ given $X = x$ is:

$$f_{\Theta|X}(\theta|x) = \frac{f(x|\theta)\,g(\theta)}{f_X(x)}$$

<u>Exercise 3.34</u> Suppose $g(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}$, $0<\theta<1$

as in 2.2.4, p. 58 (the beta density with parameters a and b). Take $a=b=3$. Then

$$g(\theta) = \frac{\Gamma(6)}{\Gamma(3)\Gamma(3)}\theta^2(1-\theta)^2 = \frac{5!}{2!\,2!}\theta^2(1-\theta)^2 = 30\,\theta^2(1-\theta)^2,\quad 0<\theta<1$$

Let now X have (discrete) freq. function $p(x|\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$, $x = 0,1,2,\dots,n$, say for $n = 10$. Then the marginal freq fcn of X is: $p_X(x) = \int_0^1 \binom{10}{x}\theta^x(1-\theta)^{10-x}\cdot 30\theta^2(1-\theta)^2\,d\theta$

$$= 30\binom{10}{x}\cdot\int_0^1 \theta^{2+x}(1-\theta)^{12-x}\,d\theta = 30\binom{10}{x}\frac{\Gamma(3+x)\Gamma(13-x)}{\Gamma(16)},$$

$x = 0,1,\dots,10$. So the posterior density of Θ given $X=x$ is:

$$\frac{\cancel{30\binom{10}{x}}\,\theta^{2+x}(1-\theta)^{12-x}}{\cancel{30\binom{10}{x}}\,\Gamma(3+x)\Gamma(13-x)}\,\Gamma(16),\quad 0<\theta<1,$$ <u>or beta with parameters:</u> $\boxed{a' = 3+x,\ b' = 13-x}$