Notes 4. Math 481/581 Mathematical Statistics I

(p.1)

Uniform Distribution on a region $R$ of the plane

If $R$ is a region of the plane with area $|R| < \infty$, then a joint density for a uniformly distributed random pair $(X, Y)$ is defined by:

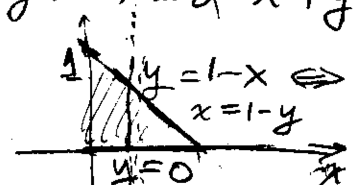
$$f(x,y) = \begin{cases} \frac{1}{|R|}, & (x,y) \in R \\ 0, & \text{else.} \end{cases}$$

($f$ is constant for $(x,y) \in R$ and zero otherwise)

Example. Let $R = \{(x,y) : x \geq 0, y \geq 0 \text{ and } x + y \leq 1\}$.

Then $|R| = \frac{1}{2} \cdot 1 \cdot 1 = 1/2$.

$$f(x,y) = \begin{cases} 2, & (x,y) \in R \\ 0 & \text{else.} \end{cases}$$




The marginal density of $X$ is obtained as:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{1-x} 2\,dy = \boxed{2(1-x),\ 0 \leq x \leq 1}$$

Similarly $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{1-y} 2\,dx = \boxed{2(1-y),\ 0 \leq y \leq 1}$

Obviously $X$ and $Y$ are not independent r.v.'s since $f(x,y) \neq f_X(x) \cdot f_Y(y)$.

Example. Let $X$ and $Y$ be independent and each uniformly distributed on $[0, T]$.

Thus $f_X(x) = \begin{cases} 1/T, & 0 \leq x \leq T \\ 0, & \text{else} \end{cases}$ and $f_Y(y) = \begin{cases} 1/T, & 0 \leq y \leq T \\ 0, & \text{else} \end{cases}$

So by independence $f(x,y) = f_X(x) f_Y(y) = \begin{cases} 1/T^2 & 0 \leq x \leq T, 0 \leq y \leq T \\ 0 & \text{else} \end{cases}$

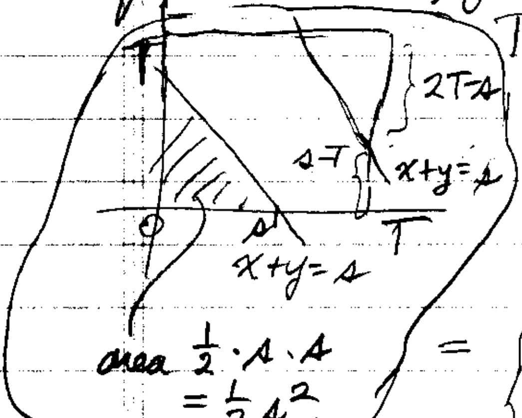
Thus the random pair $(X, Y)$ is uniformly distributed on the square $0 \leq x \leq T$, $0 \leq y \leq T$.

## Notes 4

(p.2) Example. Let $X$ and $Y$ be independent with each uniformly distributed on $[0, T]$. Find the cumulative distribution function of $S := X + Y$.

We want to calculate $P(S \le s)$ for each $s \in [0, 2T]$. Since $(X, Y)$ has density $f(x,y) = 1/T^2$, $0 \le x \le T$, $0 \le y \le T$ and $f(x,y) = 0$, else, and since $P(X + Y \le s) = \iint_{x+y \le s} f(x,y)\,dx\,dy$,

we integrate the constant $1/T^2$ over the part of the square $0 \le x, y \le T$ such that $x + y \le s$.




Thus $P(S \le s) = P(X + Y \le s)$

$$= \begin{cases} \dfrac{\frac{1}{2} s^2}{T^2}, & 0 \le s \le T \\ 1 - \dfrac{\frac{1}{2}(2T - s)^2}{T^2} & T \le s \le 2T \end{cases}$$

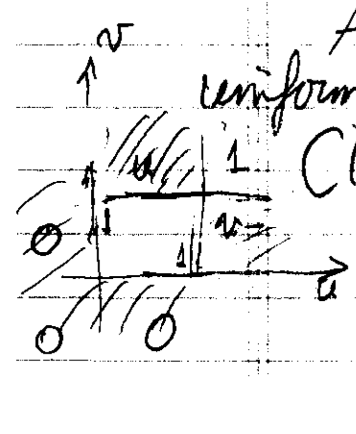
$$= \begin{cases} \frac{1}{2}\left(\frac{s}{T}\right)^2 & 0 \le s \le T \\ 1 - \frac{1}{2}\left(2 - \frac{s}{T}\right)^2 & T \le s \le 2T. \end{cases}$$

The density of $S$ is therefore $\frac{d}{ds} P(S \le s)$ or

$$f_S(s) = \begin{cases} s/T^2 & 0 \le s \le T. \\ \left(2 - \frac{s}{T}\right)\left(\frac{1}{T}\right), & T \le s \le 2T. \end{cases}$$

### Copula.

A copula is a joint distribution with uniform marginal distributions. The following is a copula for $|\alpha| \le 1$:




$$C(u,v) = \begin{cases} u v \left(1 + \alpha(1-u)(1-v)\right), & 0 \le u \le 1, 0 \le v \le 1, \\ 1 & \text{where } u \ge 1 \text{ and } v \ge 1, \\ 0 & \text{where } u \text{ or } v \le 0, \\ u, & \text{where } 0 \le u \le 1 \text{ and } v \ge 1, \\ v, & \text{where } 0 \le v \le 1 \text{ and } u \ge 1. \end{cases}$$

(over)

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(p.3) We have that $C(u,v)$ is a joint cumulative distribution function of a pair $U, V$ because

$C(U,V)$ is non-decreasing and continuous. Indeed we have a joint density $\boxed{c(u,v) := \frac{\partial^2 C}{\partial u \partial v}(u,v) = \alpha(1-2u)(1-2v)+1, \quad 0 \le u,v \le 1}$ that is non-negative since $|1-2u| \le 1$ and $|1-2v| \le 1$ for $0 \le u,v \le 1$ and since $|\alpha| \le 1$ by assumption.

Furthermore $C(u,\infty) := P(U \le u, V < \infty)$

$$= P(U \le u) = \begin{cases} 0 & u \le 0 \\ u & 0 \le u \le 1 \\ 1 & u \ge 1 \end{cases}$$

Similarly $C(\infty, v) = P(V \le v) = \begin{cases} 0 & v \le 0 \\ v & 0 \le v \le 1 \\ 1 & v \ge 1 \end{cases}$,

while $C(\infty,\infty) := \lim_{u,v \to \infty} C(u,v) = 1$.

Thus $C$ is a joint cdf. with uniform marginals.

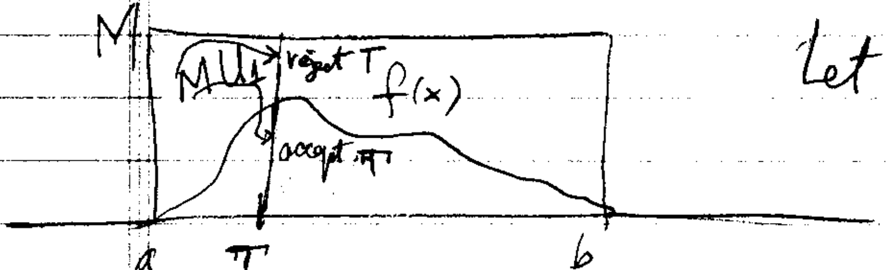
In general, if $F(x)$ and $G(y)$ are distribution functions Then $H(x,y) = F(x)G(y)\{1 + \alpha(1-F(x))(1-G(y))\}$. is a joint cdf. with marginals $F$ and $G$ whenever $|\alpha| \le 1$. The density of $H(x,y)$ is by the chain rule: $h(x,y) = c(F(x), G(y)) \cdot f(x) g(y)$ when $F(x)$ and $G(y)$ have densities $f(x)$ and $g(y)$ respectively.

Note that $c(u,v)$ is not constant unless $\alpha = 0$ so only in the case $\alpha = 0$ does the copula correspond to the case of a joint distribution of independent random variables

Note 4

(p. 4) <u>Rejection Method</u> <u>for Simulating a random Variable.</u>

Let $f$ be a density that is non-zero only on an interval $[a,b]$, and let $M$ be a constant $\geq f(x)$, $x \in [a,b]$.




Let $m(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{else} \end{cases}$

be the density of a Uniform r.v. on $[a,b]$

We first generate $T$ with density $m(x)$.

( $T = a + (b-a)U_0$ for $U_0 \sim$ Uniform$[0,1]$ ).

Second generate $U_1 \sim$ Uniform$[0,1]$ independent of $T$

If $MU_1 \leq f(T)$ then let $X := T$ (accept $T$), otherwise reject $T$ and go through these two steps again.

<u>Claim</u> $X$ has density $f$.

$$P(X \in [x, x+dx]) = P(T \in [x, x+dx] \mid \text{accept } T)$$

$$= P(T \in [x,x+dx] \text{ and accept } T) / P(\text{accept } T)$$

$$= \frac{P(MU_1 \leq f(T) \text{ and } T \in [x, x+dx])}{P(MU_1 \leq f(T) \text{ and } T \in [a,b])}$$

$$= \frac{(f(x)/M) \cdot \left(\frac{dx}{b-a}\right)}{\int_a^b \frac{f(x)}{M} \frac{dx}{b-a}} = \frac{f(x)\,dx}{\int_a^b f(x)\,dx} = \frac{f(x)\,dx}{1}$$

$$= f(x)\,dx \quad \text{QED}$$