Review Problems on Chapters 1-3   Modern Analysis I   Morrow   Fall, 2005

Text: Fitzpatrick

1. (a) Define what is meant by a bounded sequence $\{a_n\}$ of real numbers.
(b) Show that $\{1 + (-1)^n/n\}$ is a bounded sequence of real numbers. Is it monotone?
(c) Consider again $a_n = 1 + (-1)^n/n$. Define $A_n := \sup\{a_k : k \geq n\}$. Compute A_1, A_2, A_3, A_4. Is $\{A_n\}$ monotone?

2. (a) Define what it means for a sequence $\{a_n\}$ of real numbers to converge.
(b) State the Monotone Convergence Theorem for sequences of real numbers.
(c) Let $a_1 = 2$, and $a_{n+1} = +\frac{1}{2}a_n + 7$ for all $n \geq 1$. Prove that $\{a_n\}$ converges, and use theory to establish the value of the limit of this sequence.

3. Let S be a bounded nonempty set of real numbers.
(a) Define what is meant by $\sup S$.
(b) Denote $b = \sup S$. Prove that there is a sequence $\{s_n\}$ of elements of S such that $\lim s_n = b$.

4. (a) Prove that there is a sequence of rational numbers that converges to $\sqrt{2}$.
(b) Prove that there is a sequence of irrational numbers that converges to 1.

5. (a) State two equivalent definitions of the continuity of a function $f : D \to \mathbf{R}$ at $x_0 \in D$.
(b) State the negation of one of these definitions and then use this definition of discontinuity to show that $f : [0, \infty) \to \mathbf{R}$ defined by $f(x) = \sin(1/x)$ if $x > 0$, and $f(0) = 0$, is discontinuous at $x = 0$.

6. Assume that $f : (-1, 1) \to \mathbf{R}$ is continuous at $x = 0$. Assume that $f(0) = 2$. Use one of the definitions of continuity to prove that there exists an interval $I = (-r, r)$ for some $r > 0$ such that for all x in I we have $f(x) > 1$.

7. Suppose that $f : [0, \infty) \to \mathbf{R}$ is continuous, $f(0) = 1$, and $0 \leq f(x) \leq 10 + x$ for all $x \geq 0$. Prove that there exists a solution of the equation $f(x) = x^2$ for some $x \in \mathbf{R}$.

8. (a) State the definition of a limit point of a subset D of the real numbers.
(b) For each set D of real numbers find its set of limit points. Prove your assertions for extra points.

(i) $D = \{-1, 1\}$
(ii) $D = \{1/n,\ n = 1, 2, 3, ...\}$
(iii) $D = \{\text{all irrational numbers}\}$

9. Show that for $f(x) = 1/x$ we have that
(i) f is uniformly continuous on $[a, \infty)$ for any fixed $a > 0$.
(ii) f is not uniformly continuous on $(0, \infty)$.

10. (a) State the definition of a compact set of real numbers.
(b) Determine which of the following are compact sets. Prove your assertions for extra points. In parts (iii) and (iv) either prove, state a theorem, or give a counterexample.

(i) $\{1,\ 2,\ 3\}$
(ii) $[1/2, 1] \cup [1/4, 1/3] \cup [1/6, 1/5] \cup \cdots \cup \{0\}$
(iii) $f([0, 1])$ where $f : [0, 1] \to \mathbf{R}$ is continuous.
(iv) $f((0, 1])$ where $f : (0, 1] \to \mathbf{R}$ is continuous.