Math 431 Modern Analysis I Fall, 2005

Chapter 1 Review

1. Prove that if the real number x satisfies (i) $x \geq 1$ and (ii) $x \leq 1 + \epsilon$ for every $\epsilon > 0$ then $x = 1$.

2. Let $\delta > 0$. Use math induction to prove that $(1+\delta)^n \geq 1 + n\delta$ for all $n = 1, 2, 3, \ldots$.

3. Let S be a non-empty set of real numbers that is bounded above. Put $b = \sup S$. Prove by induction that for each $n = 1, 2, 3 \ldots$, there is an element $y_n \in S$ such that the following both hold for each such n:

$$\text{(i) } y_{n+1} \geq y_n, \text{ and } \text{(ii) } y_n > b - 1/n.$$

4. By Theorem 1.11, for each $n = 1, 2, 3, \ldots$ there is a rational number $r_n \in (\sqrt{2}, \sqrt{2} + 1/n)$. For each n make such a choice of a rational and then let S be the set of all the rational numbers r_n so obtained. Prove that $\inf S = \sqrt{2}$.

5. Let $c > 1$, and inductively define $c_1 = c$ and $c_{n+1} = \sqrt{c_n}$ for each $n = 1, 2, 3, \ldots$.
(a) Prove that $c_{n+1} < c_n$ for each n.
(b) Prove that $\inf\{c_n : n = 1, 2, 3, \ldots\} = 1$.

6. Prove that if y_1 and y_2 are real numbers that satisfy $y_1 \geq 2$ and $|y_2 - y_1| \leq 1$, then $y_1 \geq 1$.

Sample Solutions

1. Proof by contradiction. Suppose $x > 1$. Then $x - 1 > 0$. Therefore by the Archimedean property, Propsition 1.9, there exists some positive integer n so that $1/n < x - 1$, and so $x > 1 + 1/n$. Therefore if we take $\epsilon := 1/n > 0$ we obtain a contradiction to our hypothesis that $x \leq 1 + \epsilon$ for every positive number ϵ. Hence $x \leq 1$. Therefore since both $x \geq 1$ and $x \leq 1$ we have $x = 1$.

2. We first check that the statement holds for $n = 1$: $(1+\delta)^1 = 1 + \delta$. So indeed the inequality does hold as an equality when $n = 1$. Next we assume that

$$(*) \quad (1+\delta)^k \geq 1 + k\delta$$

for some $k \in \mathbb{N}$. We must show that (*) holds with $k+1$ in place of k. We write $(1+\delta)^{k+1} = (1+\delta)(1+\delta)^k \geq (1+\delta)(1+k\delta)$ by (*). But this last expression is simply

$$1 + \delta + k\delta + k\delta^2 > 1 + (k+1)\delta$$

Thus the inductive step is completed. Q.E.D.

3. By the definition of the supremum as a least upper bound we know that b is an upper bound of S and that any number a with $a < b$ is not an upper bound of S. Thus given $a < b$ there is an element $y \in S$ with $y > a$. Hence there is an element $y_1 \in S$ with $y_1 > b - 1$. If $y_1 < b$ then there exists $y_2 \in S$ with $y_2 > \min\{b - 1/2, y_1\}$. If $y_1 = b$ then $b \in S$ and we take $y_2 = b$ as well. Therefore both (i) and (ii) hold for these choices of y_1 and y_2. We proceed by induction to find y_n for all remaining positive integers $n \geq 3$. Our induction hypothesis is that $y_1 \leq y_2 \leq \cdots \leq y_k \leq b$ with $y_i \geq b - 1/i$ for all $i = 1, 2, \ldots k$. We then construct y_{k+1} as before: if $y_k < b$ then there exists $y_{k+1} \in S$ with $y_{k+1} > \min\{b - 1/(k+1), y_k\}$. If $y_k = b$ then $b \in S$ and we take $y_{k+1} = b$ as well. Then it is easy to see that given that the induction hypothesis holds as stated for some integer $k \geq 2$ then it continues to hold for $k+1$ in place of k. Therefore by mathematical induction (i) and (ii) hold for all $n \geq 1$.

4. We must show that $\inf S = \sqrt{2}$. We therefore must show (1): $\sqrt{2}$ is a lower bound for S, and (2): if $a > \sqrt{2}$ then a is not a lower bound of S. The statement (1) is immediate since each element of S is constructed to be larger than $\sqrt{2}$. Next, assume that there exists a number $a > \sqrt{2}$ that is a lower bound of S. Then $a - \sqrt{2} > 0$. Thus by the Archimedean property there is some $n \in \mathbb{N}$ such that $1/n < a - \sqrt{2}$. Hence by construction of r_n there is $r_n \in S$ with $r_n < \sqrt{2} + 1/n < a$. This is a contradiction to the assumption that a is a lower bound of S. Hence there is no lower bound that is larger than $\sqrt{2}$. Hence (2) is establshed. Q.E.D.

5. (a) Let $c_1 = c > 1$. We fist show that $c_2 := \sqrt{c_1} < c_1$. Indeed, $c_1 > \sqrt{c_1}$ if and only if $\sqrt{c_1}(\sqrt{c_1} - 1) > 0$. But we have that $c_1 > 1$ implies that $\sqrt{c_1} > 1$ (else we get a contradiction), so the factor $(\sqrt{c_1} - 1) > 0$. Hence indeed $c_2 < c_1$. Next we make the induction hypothesis as follows. For some $k \geq 2$ there holds

$$(\mathrm{P}_k): \quad c_1 > c_2 > \cdots > c_k > 1$$

We have shown that (P_2) holds. We must show that (P_k) implies (P_{k+1}). Given (P_k) we write $c_{k+1} := \sqrt{c_k}$. Since $c_k > 1$ we have by the very same argument as before that $c_k > c_{k+1} > 1$. Hence the inductive step in completed. Part (a) is proved.
(b) Since S is bounded below by 1 we have that the infimum exists and $a := \inf S \geq 1$. We show that the infimum is a=1 by contradiction. Suppose therefore that $a > 1$. By the definition of $\inf S$ there is some element of S, say $s = c_n$ that satisfies $s < a + (a-1)/2 = 1 + (3/2)(a-1)$. Write $s = 1 + \delta$ with some $(3/2)(a-1) > \delta > 0$. It is not difficult to show in general (by squaring both sides of the following inequality) that for all $\delta > 0$, we have $\sqrt{s} = \sqrt{1+\delta} < 1 + \delta/2$. Therefore, since $\delta < (3/2)(a-1)$ we find that $c_{n+1} = \sqrt{s} < 1 + \delta/2 < 1 + (3/4)(a-1) < a$. Therefore a is not a lower bound of S. This is a contradiction to our assumption that the greatest lower bound a satisfies $a > 1$. Hence $a \leq 1$. Since also $a \geq 1$ we have proved $a = 1$. Q.E.D.

6. We have that $|y_2 - y_1| \leq 1$ means $-1 \leq y_2 - y_1 \leq 1$. Therefore $-1 + y_1 \leq y_2 \leq 1 + y_1$. Hence, in particular, since $y_1 \geq 2$ we have that $1 = -1 + 2 \leq -1 + y_1 \leq y_2$. Q.E.D.