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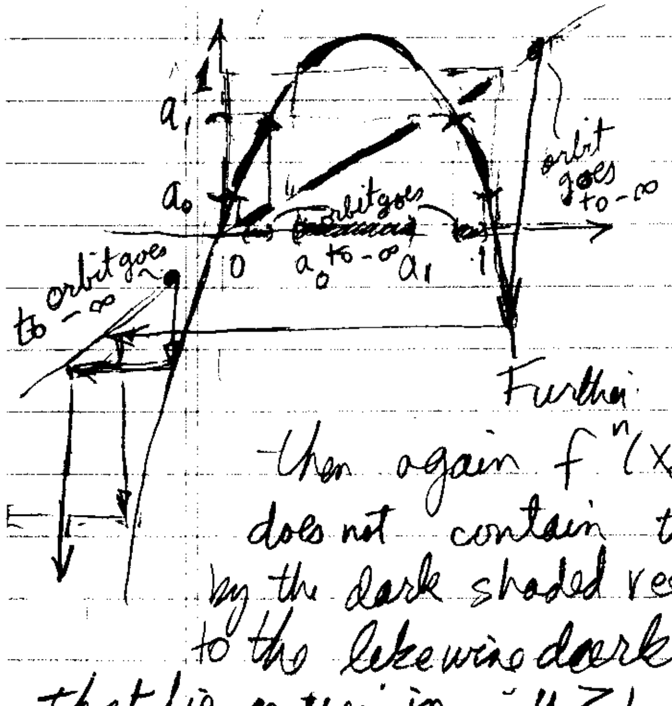
Notes 6. Chaos.

(p.1) **Introduction to Chaotic maps.**

Given $f: I \to \mathbb{R}$, I an interval of $\mathbb{R}$, we study the set $X \subset I$ such that for $x_0 \in X$ the orbit $\mathcal{O}(x_0) = \{f^n(x_0) : n \geq 0\}$ remains in a certain finite interval $[a,b]$. Thus in particular $X \subset I$; but depending on the map f, X may not be an interval itself.

Examples ① If $f: [a,b] \to [a,b]$ then all the orbits remain in $[a,b]$ so we have $X = [a,b]$.

② Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 5x(1-x)$. By graphical analysis we have that for $x_0 < 0$ and $x_0 > 1$ $f^n(x_0) \to -\infty$



Further if $x \in (a_0, a_1)$ for the two solutions a_0, a_1 of $f(x) = 1$ then again

$$f^n(x_0) \to -\infty$$

Further if x_0 is such that $y = f(x_0) \in (a_0, a_1)$ then again $f^n(x_0) \to -\infty$. So already X does not contain the intervals of $(0,1)$ represented by the dark shaded regions along the line $y = x$ corresponding to the likewise darkly shaded portions of the curve that lie either in $y > 1$ or $y \in (a_0, a_1)$. We have that the orbit of $x_0 \in (a_0, a_1)$ escapes $[0,1]$ after 1 iteration, while for x_0 belonging to the other two intervals shown, the orbit escapes $[0,1]$ after 2 iterations.

Notes to Chaos

(p.2) If instead we take the function $h: \mathbb{R} \to \mathbb{R}$ defined on p. 102 of text (See problems 3.2 #8–#10) then this construction for X leads to the Classical Cantor set $X \subset [0,1]$. (exercise 3.2 #10).

If there is sufficient stretching and folding in $f: I \to \mathbb{R}$ such that the map $f: X \to X$ is transitive (see Definition 3.5, p. 107) then it generally happens that $f: X \to X$ will be "chaotic". This entails two features. First there will be internal order in as much as the periodic points of $f: X \to X$ will be dense in X, that is for any open interval U no matter how small with $U \cap X \neq \emptyset$, there is a point $\bar{x} \in X$ of a cyclic orbit belonging to U. Second, a typical point $x_0 \in X$ (chosen at random) will not belong to this dense set of periodic points but instead the orbit of x_0 will be unpredictable and will itself be a dense orbit: given any small open interval U with $U \cap X \neq \emptyset$ then $f^n(x_0) \in U$ will occur for some one n and in fact infinitely many n.

By definition 3.5 a map $f: X \to X$ is transitive means in particular that if U and V are any given open intervals meeting X then $f^n(U) \cap V \neq \emptyset$ for some n. If $f: [0,1] \to [0,1]$ is given by $f(x) = \mu x(1-x)$ for some $0 < \mu < \mu_\infty \approx 3.57$, then f is not transitive. In this parameter range there is for each μ a single attracting cycle $\{\bar{x}_1, \ldots, \bar{x}_k\}$. The immediate basins of attraction $J_1, \ldots, J_k$ of these fixed points under f^k will simply be mapped into one another by f. Further $f^{nk+r}(J_1) \xrightarrow[n\to\infty]{} \{\bar{x}_{r+1}\}$. So $f^n(J_1)$ does not "go all over" the space.