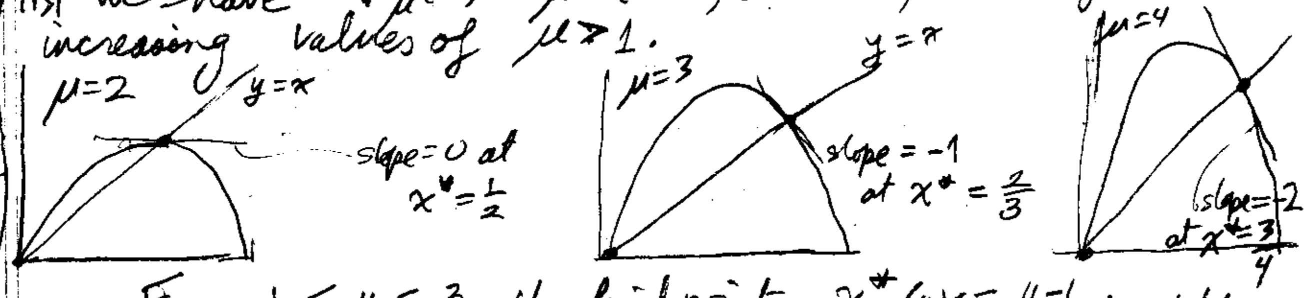
~~Notes~~ 3 Chaos.

(p.1) Period doubling route to Chaos.

To understand the bifurcation diagram in (text) figures 1.25 and 1.26 for the logistic family, we first show how figure 1.25 arises.

First we have $F_\mu(x) = \mu x(1-x)$, $0 \le x \le 1$, shown for increasing values of $\mu > 1$.

Fixed points $\boxed{F_\mu(x) = x \iff x = 0 \text{ or } \mu = \mu x + 1. \quad x^*_-(\mu) = 0, \quad x^*_+(\mu) = \frac{\mu-1}{\mu}}$



For $1 < \mu < 3$ the fixed point $x^*_+(\mu) := \frac{\mu-1}{\mu}$ is stable.

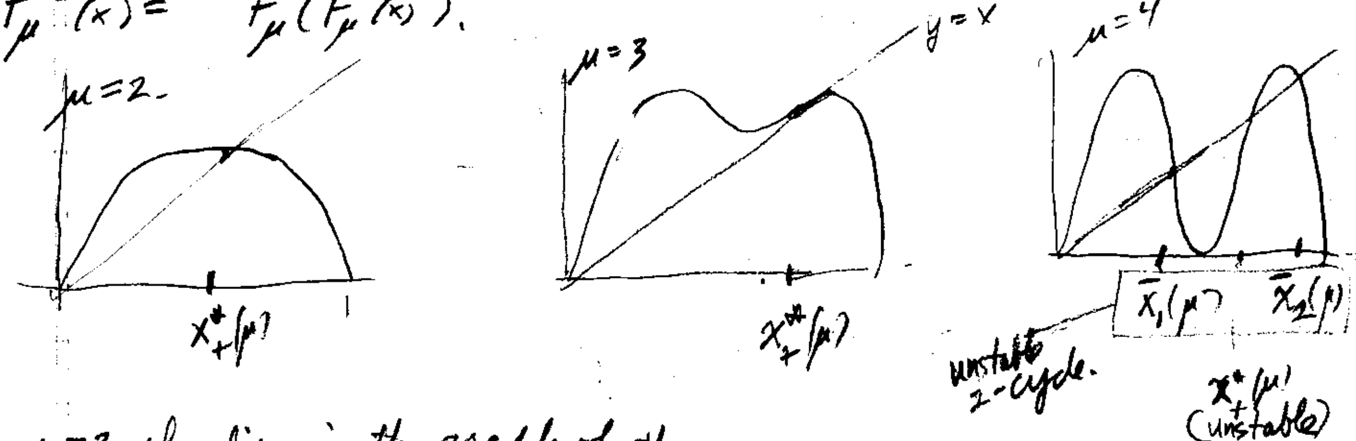
Now $F_\mu'''(x) = 0$, but $F_\mu''(x) = -2\mu \neq 0$

so $SF_\mu(x) = 0 - \frac{3}{2}\left(\frac{-2\mu}{\mu - 2\mu x}\right)^2 < 0$. Thus by Thm 1.5

for $\mu = 3$, $x^*_+(\mu) = 2/3$ is an asymptotically stable fixed point

Next, for $\mu > 3$ $x^*_+(\mu)$ is unstable by Thm 1.3 since $F_\mu'(x^*_+(\mu)) < -1$.

But we now study a sequence of graphs of the 2nd iterates $F_\mu^2(x) = F_\mu(F_\mu(x))$.



For $\mu > 3$ the dip in the graph of the 2nd iterate begins to deepen such that a 2-cycle appears for $\mu > 3$. The 2-cycle is unstable by $\mu = 4$. To determine for which μ the 2-cycle becomes unstable one can solve $F_\mu'(\bar{x}_1(\mu)) F_\mu'(\bar{x}_2(\mu)) = -1$ for the explicit solution $\bar{x}_1(\mu), \bar{x}_2(\mu)$ given in (1.30).

<u>Notes 3</u> <u>Chaos</u>.

(p.2) Notice that at $\mu = 3$ we have $F'(x^*) = F_3'(x^*) = -1$, for $x^* = x_+^*(3)$. Next by calculation of F^2! as before, $F^{2\prime}(x) = \frac{d}{dx} F(F(x)) = F'(F(x))F'(x)$, and, since $F(x^*) = x^*$ we have $F^{2\prime}(x^*)$ $= F'(F(x^*))\, F'(x^*) = F'(x^*)^2 = (-1)^2 = 1$. Further at $\mu = 3$ again, using $F^{2\prime\prime}(x) = F'(x)^2 F''(F(x)) + F'(F(x))F''(x)$ we have $F^{2\prime\prime}(x^*) = (-1)^2 \cdot 1 + (-1)\cdot 1 = 0$. So we have a point of inflection on the graph of F_3^2 at x^*. Now the "well" in the graph of F_μ^2 becomes deeper as μ increases past $\mu = 3$ while there remains a fixed point $x^* = \frac{\mu-1}{\mu}$ at which $F_\mu'(x^*) > 0$, we must have that two additional fixed points $\bar{x}_1, \bar{x}_2$ relative to F_μ^2 appear for $\mu > 3$. See for example (p.5) of Notes 2 where the graph of F^2 is shown for $\mu = 3.25$. It follows that since F has no other fixed points that $\bar{x}_1, \bar{x}_2$ is a 2 cycle for F. Indeed $F^2(\bar{x}_1) = \bar{x}_1$ and $F^2(\bar{x}_2) = \bar{x}_2$ while $F(\bar{x}_1) \neq \bar{x}_1$ and $F(\bar{x}_2) \neq \bar{x}_2$, so $\{F(\bar{x}_1), \bar{x}_1\}$ is a two cycle because $F(F(\bar{x}_1)) = \bar{x}_1$. By the same reasoning $\{\bar{x}_2, F(\bar{x}_2)\}$ is also a 2-cycle. If these are not the same 2-cycles then we would have even more fixed points on the graph of F^2, an impossibility by counting the degree of F^2. (F^2 is of degree 4 and there are already 4 fixed points; the maximum number possible.) So we have an emergent two cycle. The common slope $F^{2\prime}(\bar{x}_1) = F^{2\prime}(\bar{x}_2)$ for μ very near $\mu = 3$ but $\mu > 3$ is very near 1 but less than 1. (over)

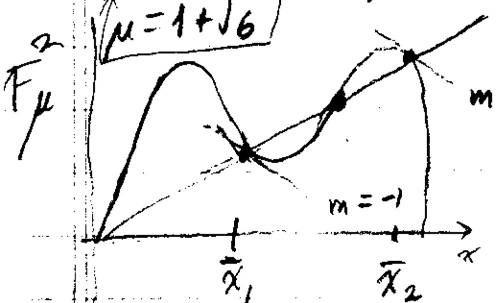
Notes 3 Chaos

(p. 3) Hence the emergent 2-cycle $\bar{x}_1(\mu)$, $\bar{x}_2(\mu)$ (shown by algebraic calculations in (1.30)) is just stable near $\mu = 3$, becomes nearly super-stable as μ increases to near 3.25 until finally, as μ continues to increase, using that

$$F_\mu^{2\,\prime}(\bar{x}_1) = F_\mu^{2\,\prime}(\bar{x}_2) = F_\mu'(\bar{x}_1)\,F_\mu'(\bar{x}_2)$$
$$= -\mu^2 + 2\mu + 4 \quad \text{(see page 39 of text)},$$

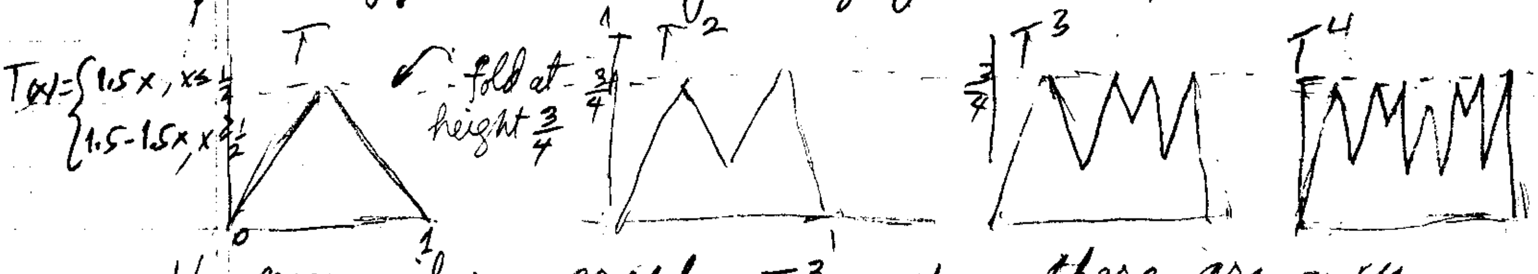
we have that

$$F_\mu^{2\,\prime}(\bar{x}_1) = -1 \quad \text{at} \quad \mu = 1+\sqrt{6} \approx 3.449$$



As μ continues to increase, the well gets deeper and the hills get steeper so the slope of F^2 at $\bar{x}_1$ and $\bar{x}_2$ dips below -1 for $\mu > 1+\sqrt{6}$. Exactly at $\mu = 1+\sqrt{6}$ one can apply the Schwarzian derivative to F^2 (see 1.8 # 1) to show stability of the 2-cycle at the crossover value of the parameter.

By analogy with the family of tent maps we have



However, if we graph $F^3_{1+\sqrt{6}}$ then there are only two fixed points, the ones we already have for $F_{1+\sqrt{6}}$.

Yet by the analysis we have made for the way the graph of F_μ^2 changes and by the same calculations made for F_μ and F_μ^2 at $\mu = 3$ and $x = x_+^*(\mu)$ we can replace F_3 by $F^2_{1+\sqrt{6}}$ and F_3^2 by $F^4_{1+\sqrt{6}}$ and $x^*(3)$ by $\bar{x}_1(1+\sqrt{6})$