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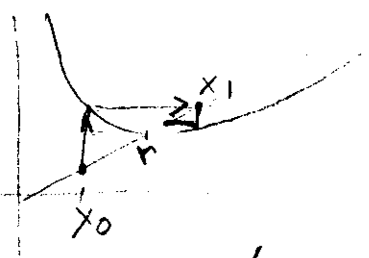
Notes 2 Chaos

(p.1)

Newton's Method

Consider solving the equation $g(x) := x^2 - 2 = 0$. The Newton's method is to construct the function
$$f(x) = x - \frac{g(x)}{g'(x)} \quad \text{(assuming } g'(r) \neq 0$$
for the root r of $g(x) = 0$). The function f is then iterated for some approximation x_0 of r. For the current example we have
$$f(x) = x - \frac{(x^2-2)}{2x} = x - \frac{x}{2} + \frac{1}{x}$$
or $f(x) = \frac{x}{2} + \frac{1}{x}$.

We pick x_0 near the root $r = \sqrt{2}$. Notice that $f(r) = r$ since $g(r) = 0$.

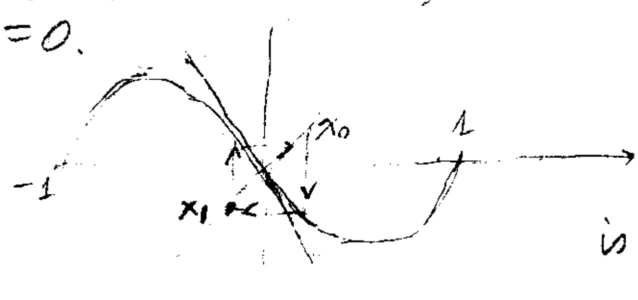


So r is a fixed point for f.
Furthermore $f'(x) = \frac{1}{2} - \frac{1}{x^2}$

So $f'(\sqrt{2}) = \frac{1}{2} - \frac{1}{2} = 0$. It is true in general for Newton's function that $f'(r) = 1 - \frac{g'(r)^2 - g(r)g''(r)}{g'(r)^2} = 0$. Thus by Theorem 1.3 r is a (super-) stable fixed point for f.

Schwarzian Derivative

Consider $f(x) = x^3 - x$, $-1 \leq x \leq 1$ with fixed point $x^* = 0$.



Now $f'(x) = 3x^2 - 1$
so $f'(0) = -1$
Hence $x^* = 0$ is a non-hyperbolic fixed pt.

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(p.2) Let $x_0 > 0$ be small. Then by positive concavity of f for $x > 0$ we have that $x_1 < 0$ but $-x_0 < x_1$. Thus $|x_1| < x_0$. By symmetry we see that $x_2 < |x_1|$. Hence $0 < x_2 < |x_1| < x_0$. Hence $|x_n|$ is decreasing. If $|x_n|$ converges to $a > 0$ then $a, -a$ must be a two cycle. Indeed if n is even then $x_n > 0$ while $x_{n+1} < 0$ and both are near a in absolute value. Hence we must have $f(a) = -a$ or $a^3 - a = -a$ or $a^3 = 0$ or $a = 0$. So there is no 2-cycle. Hence indeed we have that $x^* = 0$ is an asymptotically stable fixed point.

This example can be handled by an analytic condition as in Thm 1.5. The hypothesis of this theorem is that $f'(x^*) = -1$ for a fixed pt x^* of f. Then one computes a Schwarzian derivative at x^* to determine if the fixed point is stable or unstable. The idea of this theorem is to consider the second iterate $g = f^2$. It turns out that if x^* is asymptotically stable for g (it is automatically fixed for g since $g(x^*) = f(f(x^*)) = f(x^*) = x^*$) then x^* will be asymptotically stable for f. Likewise if x^* is unstable for g then it is unstable for f. Indeed the orbit of x_0 under g is $x_0, x_2, x_4, \ldots$, where x_{2n}, are the even indexed terms in the orbit of x_0 under f. But, if $x_{2n} \to x^*$ then also $x_{2n+1} = f(x_{2n}) \to f(x^*) = x^*$ by continuity. Therefore the whole sequence $x_0, x_1, x_2 \ldots$ converges to x^*.
Based on this sketch we proceed to check on stability of x^* for g.

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(p.3) We find that $g'(x) = \frac{d}{dx} f(f(x)) = f'(f(x)) f'(x)$

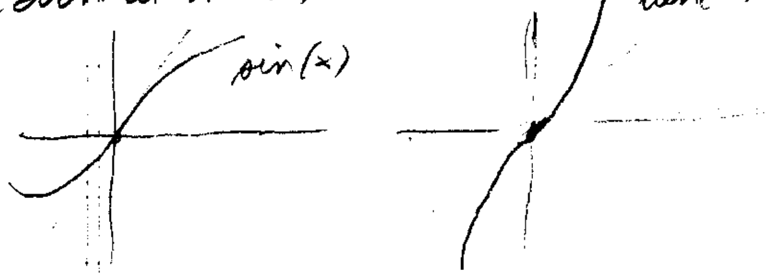
So $g'(x^*) = f'(x^*) f'(x^*) = (-1)(-1) = 1$.

Next $g''(x) = f''(f(x)) \cdot f'(x) \cdot f'(x) + f'(f(x)) f''(x)$

So $g''(x^*) = f''(x^*) (-1)(-1) + f'(x^*) f''(x^*)$

$$= f''(x^*) - f''(x^*) = 0$$

Finally we determine whether there is a change in concavity from positive to negative as in the $g(x) = \sin(x)$ example or vice versa as in the case of $g(x) = \tan(x)$ (both at $x = 0$)



We have $g'''(x)$
$= f'''(f(x)) f'(x)^3$
$+ f''(f(x)) 2 f'(x) f''(x)$
$+ f''(f(x)) \cdot f'(x) f''(x)$
$+ f'(f(x)) f'''(x)$.

Therefore $g'''(x^*) = f'''(x^*) (-1)^3 - 2 f''(x^*)^2$
$- f''(x^*) - f'''(x^*)$

$$= -2 f'''(x^*) - 3 f''(x^*)^2$$

$$= 2\left[\frac{f'''(x^*)}{f'(x^*)} - \frac{3}{2}\left[\frac{f''(x^*)}{-f'(x^*)}\right]^2\right] = 2Sf(x^*)$$

(since $f'(x^*) = -1$)

If $g'''(x^*) < 0$ (as in $g(x) = \sin(x)$ example) we have that x^* is an asymptotically stable fixed point.

If $g'''(x^*) > 0$ (as in $g(x) = \tan(x)$ example) then x^* is unstable.

Applied to our example above we have $f(x) = x^3 - x$.

Therefore $Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2$

$= \frac{6}{3x^2-1} - \frac{3}{2}\left(\frac{6x}{3x^2-1}\right)^2$ So $Sf(0) = -6 - \frac{3}{2}(0)^2 = -6 < 0$

Thus by the Theorem 1.5, $x^* = 0$ is asymptotically stable for f.