Math 313.002 Linear Algebra Review for Test2 Fall 2007

1. Show that the range of the linear operator defined by the equations is not all of $\mathbb{R}^3$, and find a vector that is not in the range.

$$\begin{aligned} w_1 &= x_1 - 2x_2 + x_3 \\ w_2 &= 5x_1 - x_2 + 3x_3 \\ w_3 &= 4x_1 + x_2 + 2x_3 \end{aligned}$$

$\det \begin{vmatrix} 1 & -2 & 1 \\ 5 & -1 & 3 \\ 4 & 1 & 2 \end{vmatrix} = \det \begin{bmatrix} 1 & -2 & 1 \\ 0 & 9 & -2 \\ 0 & 9 & -2 \end{bmatrix} = \det \begin{bmatrix} 1 & -2 & 1 \\ 0 & 9 & -2 \\ 0 & 0 & 0 \end{bmatrix} = 0.$

So Range of $\bar{W} = A\bar{X}$ is not all of $\mathbb{R}^3$.

$\bar{w} \in$ Range : $\left[\begin{array}{ccc|c} 1 & -2 & 1 & w_1 \\ 5 & -1 & 3 & w_2 \\ 4 & 1 & 2 & w_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 1 & w_1 \\ 0 & 9 & -2 & w_2 - 4w_1 \\ 0 & 9 & -2 & w_3 - 5w_1 \end{array}\right]$

$\to \left[\begin{array}{ccc|c} 1 & -2 & 1 & w_1 \\ 0 & 9 & -2 & w_2 - 4w_1 \\ 0 & 0 & 0 & w_3 - 5w_1 - (w_2 - 4w_1) \end{array}\right]$

$(= w_3 - w_1 - w_2)$

$\bar{w} \in$ Range $\iff w_3 - w_1 - w_2 = 0$

$\bar{w} \notin$ Range $\iff w_3 - w_1 - w_2 \neq 0$.

So for example $(2,1,1) \in$ Range but $(2,1,2) \notin$ Range

2. Which of the following subsets W of P_3 (the polynomials of degree 3 or less) are in fact subspaces of P_3? Why or why not?

(a) all polynomials $ax + 1$ where a is a real number.

(b) all polynomials $ax^2 + bx + c$ where $a + b = 0$ and c is an arbitrary real number.

(c) all polynomials $e^a x$ where a is a real number.

(d) all polynomials $a + ax + bx^3$ where a and b are real numbers.

(a) $W = \{ax + 1\}$ is <u>not</u> a subspace

$c(ax+1) = cax + c \neq AX + 1$ for $c \neq 1$.

(b) $W = \{ax^2 + bx + c : a + b = 0, c \in \mathbb{R}\}$ <u>is</u> a subspace

$W = \{a(x^2) + (-a)x + c\} = \{a(x^2 - x) + c\}$

$= \text{Span}\{(x^2 - x), 1\}.$

(c) $\{e^a x\}$ is <u>not</u> a subspace because $-e^a x \neq e^A x$ for any real A. (when a is real)

(d) $\{a + ax + bx^3\} = \{a(1+x) + bx^3\} = \text{Span}\{(1+x), x^3\}$.

<u>is</u> a subspace.

3. (a) Express $\mathbf{w} = (4,3,2)$ as a linear combination of $\mathbf{u} = (1,0,-1)$ and $\mathbf{v} = (2,1,0)$.
(b) Is it possible to express any vector in $\mathbb{R}^3$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$? Why or why not?

$$\begin{array}{ccc} k_1 & k_2 & \\ \end{array}\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k_1 \; k_2$$

$$\left[\bar u \,\vdots\, \bar v \,\middle|\, \bar w\right] = \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 3 \\ -1 & 0 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

$k_2 = 3$, $k_1 = 4 - 2k_2 = 4 - 6 = -2$. So $\boxed{\bar w = -2\bar u + 3\bar v.}$

(b) no, because $\dim(\mathbb{R}^3) = 3$ while $\dim \operatorname{Span}\{\bar u, \bar v\} = 2$.

In particular $\bar w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \operatorname{Span}\{\bar u, \bar v\} \iff \left[\begin{array}{cc|c} 1 & 2 & w_1 \\ 0 & 1 & w_2 \\ -1 & 0 & w_3 \end{array}\right]$ is consistent

$\iff \left[\begin{array}{cc|c} 1 & 2 & w_1 \\ 0 & 1 & w_2 \\ 0 & 2 & w_3 + w_1 \end{array}\right]$ is consistent $\iff w_3 + w_1 = 2w_2$. This property is satisfied by $(4,3,2)$ but not by, say, $(4,3,3)$

4. Determine whether the following vectors span $\mathbb{R}^3$.
$\mathbf{v}_1 = (1,4,7)$, $\mathbf{v}_2 = (2,5,8)$, $\mathbf{v}_3 = (3,6,10)$.

$$\det\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} = \det\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{bmatrix} = -\det\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & -6 & -11 \end{bmatrix}$$

$$= -\det\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 1 \end{bmatrix} = -3 \neq 0.$$ Therefore

$A = [v_1 \,\vdots\, v_2 \,\vdots\, \bar v_3]$ is invertible so in particular any vector $\bar w \in \mathbb{R}^3$ can be written as

$\bar w = k_1 \bar v_1 + k_2 \bar v_2 + k_3 \bar v_3$ for some k_1, k_2, k_3

because in fact $\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = A^{-1}\bar w$ does the job

5 (a) Show that $\mathbf{v}_1 = (1,2,3)$, $\mathbf{v}_2 = (4,5,6)$, and $\mathbf{v}_3 = (7,8,9)$ are linearly dependent in $\mathbb{R}^3$.
(b) Express each vector as a linear combination of the other two.

Do part (b) first. This shows part (a).

$$[v_1 \vdots v_2 \vdots v_3] = \begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix} \to \begin{bmatrix}1&4&7\\0&-3&-6\\0&-6&-12\end{bmatrix} \to \begin{bmatrix}1&4&7\\0&1&2\\0&0&0\end{bmatrix} \to \begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\end{bmatrix}.$$

$= [w_1 \vdots w_2 \vdots w_3]$, say. Obviously $w_3 = -w_1 + 2w_2$.

So (*) $\boxed{v_3 = -v_1 + 2v_2}$. Hence v_1, v_2, v_3 are dependent.

Also $\boxed{v_1 = 2v_2 - v_3}$ (by solving (*) for v_1) and also $\boxed{v_2 = \frac{v_3 + v_1}{2}}$

6. Determine bases of each of the following subspaces of $\mathbb{R}^4$. State the dimension of each subspace.
(a) all vectors of the form $(a, a+b, 0, 0)$.
(b) all vectors of the form (a, b, b, b).
(c) all vectors of the form (a, a, b, c) where $a + b + c = 0$.

(a) $(a, a+b, 0, 0) = a(1,1,0,0) + b(0,1,0,0)$

Basis $\{(1,1,0,0), (0,1,0,0)\}$ dim. $= 2$

(b) $(a, b, b, b) = a(1,0,0,0) + b(0,1,1,1)$, $a, b \in \mathbb{R}$.

Basis $\{(1,0,0,0), (0,1,1,1)\}$. dim. $= 2$.

(c) $(a, a, b, c) = (a, a, b, -a-b)$ $a, b \in \mathbb{R}$.

with $a+b+c=0$

$= a(1,1,0,-1) + b(0,0,1,-1)$. dim $= 2$

Basis $= \{(1,1,0,-1), (0,0,1,-1)\}$.

7. Suppose A is a 3×5 matrix.
(a) Are the columns of A linearly independent, linearly dependent, or can you tell?
(b) What is the smallest the nullity of A can be? Explain.
(c) If the column space of A is a plane through the origin in $\mathbb{R}^3$, what are the rank and nullity of A? Explain.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{15} \\ a_{21} & a_{22} & \cdots & a_{25} \\ a_{31} & a_{32} & \cdots & a_{35} \end{bmatrix}_{3\times 5}$$

(a) $\text{Rank}(A) \leq 3$. Thus in particular, $\text{Rank}(A) < 5$. So columns of A are <u>linearly dependent</u>

(b) nullity $\geq 5-3=2$. There will be at least 2 free variables in the solution of $A\bar{x} = \bar{0}^{3\times 1}$ since there can be at most 3 leading 1's.

(c) Column space is a plane means $\text{Rank}(A) = 2$ (= dimension of a plane)
So nullity $= 5-2 = 3$

8. Let the inner product on $\mathbb{R}^2$ be given $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 4u_2v_2$ for $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$. Let $\mathbf{u} = (1,1)$ and $\mathbf{v} = (0,2)$.
Compute:
(a) $\|\mathbf{u}\|$,
(b) $\|\mathbf{v}\|$,
(c) the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$, and,
(d) the distance $d(\mathbf{u}, \mathbf{v})$.

(a) $\|u\| = \sqrt{\langle \bar{u}, \bar{u} \rangle} = \sqrt{1\cdot 1\cdot 1 + 4(1\cdot 1)} = \sqrt{5}$

(b) $\|v\| = \sqrt{\langle v, v \rangle} = \sqrt{1\cdot 0\cdot 0 + 4\cdot 2\cdot 2} = \sqrt{16} = 4.$

(c) $\cos\theta = \dfrac{\langle u, v \rangle}{\|u\| \, \|v\|} = \dfrac{1\cdot 1\cdot 0 + 4\cdot 1\cdot 2}{\sqrt{5}\cdot 4} = \dfrac{8}{\sqrt{5}\cdot 4}$
$= 2/\sqrt{5}$

(d) $d(u,v) = \|\bar{v} - \bar{u}\| = \left\| \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\|$

$= \sqrt{\left\langle \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\rangle} = \sqrt{1(-1)(-1) + 4(1)(1)} = \sqrt{5}.$

9. let $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ -2 & -3 & -5 & -1 \\ 1 & 1 & 2 & 0 \end{bmatrix}$.

(a) Find a basis for the row space of A.

(b) Find a basis for the nullspace of A.

(c) Verify that every vector in the row space is orthogonal to every vector in the nullspace (for the Euclidean inner product =dot product on $\mathbb{R}^4$).

$$A \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for rowspace = $\{(1,2,3,1) \quad (0,1,1,1)\}$

nullspace. $x_4 = t$

$x_3 = s$

$x_2 = -s-t$

$x_1 = -2x_2 - 3s - t = -2(-s-t) - 3s - t$

$= -s + t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{matrix} -s+t \\ -s-t \\ s \\ t \end{matrix} = s\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Basis for nullspace $\left\{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}\right\}$.

$(1,2,3,1) \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} = 0$

$(1,2,3,1) \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} = 0$

$(0,1,1,1) \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} = 0$

$(0,1,1,1) \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} = 0$.