2.5 The Matrix Inverse.

Definition Given a square matrix $A^{n\times n}$ we say A is invertible if there is a square matrix A^{-1} such that $AA^{-1} = A^{-1}A = I^{n\times n}$.

Example. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Call an unknown matrix $X = \begin{bmatrix} x & u \\ y & v \end{bmatrix}$ to stand in for A^{-1} if A^{-1} exists. Then $AX = I$ means

$$A \left[\begin{array}{c|c} x & u \\ y & v \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array}\right].$$ So in particular $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

and $A\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So we must solve two systems simultaneously with the same matrix A of coefficients. We do this by reducing the doubly augmented matrix.

$$\left[\begin{array}{cc|c|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow[R2=R2-R1]{} \left[\begin{array}{cc|c|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

To read off the solution in an easy way, we continue to reduce the whole augmented matrix:

$$\xrightarrow[R1=R1-R2]{} \left[\begin{array}{cc|c|c} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

Therefore we have $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\boxed{X = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}$

Hence we have found X so that $AX = I$. We also check easily that $XA = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$. So $X = A^{-1}$.

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Let's review the method by which we produced A^{-1} above by introducing elimination matrices.

For $E_{21} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ we have $E_{21}A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$,

and then following by $E_{12} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ we arrive at $E_{12}E_{21}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Therefore $M = E_{12}E_{21}$ is a left inverse of A.

But we already found a right inverse X so that $AX = I$.

These left and right inverses must be the same as follows:

$$M = MI = M(AX) = (MA)X = IX = X.$$

We have the <u>result</u> that: $A^{n \times n}$ <u>is invertible</u> <u>if and only if</u> A has n <u>pivots</u> in the row reduction, <u>or</u> <u>because</u> A is <u>square</u>, <u>if and only if</u> A <u>may be reduced after appropriate row multiplications</u> to the <u>identity matrix</u>. Therefore schematically we have

$$[A \mid I] \xrightarrow[\text{"row reduction"}]{} [I \mid A^{-1}] \quad \text{whenever } A \text{ is invertible.}$$

<u>Example</u>. <u>A non-invertible case</u>. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 1 \\ 2 & 0 & 4 \end{bmatrix}$. Show A^{-1} does not exist.

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 2 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R2=R2-R1]{} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -4 & -2 & -1 & 1 & 0 \\ 2 & 0 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R3=R3-2R1]{} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -4 & -2 & -1 & 1 & 0 \\ 0 & -4 & -2 & -2 & 0 & 1 \end{array}\right] \xrightarrow[R3=R3-R2]{} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -4 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

Since A does not have 3 pivots, A is <u>not</u> <u>invertible</u>. no 3rd pivot.

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Inverse of a product of invertible matrices

Suppose that $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ with $A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

and $B = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ with $B^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

What is $(AB)^{-1}$?

Claim $(AB)^{-1} = B^{-1}A^{-1}$. Indeed, $(B^{-1}A^{-1})(AB)$

$= B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$. So

$B^{-1}A^{-1}$ is a left inverse of AB, so is an inverse of AB.

Therefore $(AB)^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}$

To check this, note that $AB = \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}$ and $\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Exercise. 2.5 #26. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$. Find elimination matrices E_{21} and E_{12} and a diagonal matrix D^{-1} that reduces A to the identity.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} \xrightarrow[R2=R2-2R1]{} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \xrightarrow[R1=R1-R2]{} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

So $E_{21} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, $E_{12} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $D^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.

Therefore $A^{-1} = D^{-1}E_{12}E_{21} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & \frac{1}{2} \end{bmatrix}$.

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Inverse of a 2x2 matrix

We first show the inverse of a special 2x2 matrix $S = \begin{bmatrix} a & b \\ 0 & \alpha \end{bmatrix}$. We have

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & \alpha & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{b}{\alpha} & \frac{1}{a} & 0 \\ 0 & 1 & 0 & 1/\alpha \end{array}\right]$$

by dividing both rows by the corresponding pivot values. Here we assume both a and d are not zero else there is no inverse (because there wouldn't be a 2^{nd} pivot).

Finally

$$\left[\begin{array}{cc|cc} 1 & b/\alpha & 1/a & 0 \\ 0 & 1 & 0 & 1/\alpha \end{array}\right] \xrightarrow[R1 = R1 - \frac{b}{\alpha} R2]{} \left[\begin{array}{cc|cc} 1 & 0 & 1/a & -\frac{b}{a\alpha} \\ 0 & 1 & 0 & 1/\alpha \end{array}\right]$$

Notice that the inverse may be written as

$$S^{-1} = \frac{1}{a\alpha}\begin{bmatrix} \alpha & -b \\ 0 & a \end{bmatrix}$$ in this special case.

For the general 2x2 we must first find the two pivots a & α:

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow[R2 = R2 - \frac{c}{a} R1]{} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & d - \frac{cb}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

Now we already know the inverse of $S := \begin{bmatrix} a & b \\ 0 & d - \frac{cb}{a} \end{bmatrix}$, with $\boxed{\alpha := \frac{ad-bc}{a}}$

namely $$\boxed{S^{-1} = \frac{1}{a(d - \frac{bc}{a})}\begin{bmatrix} d - \frac{bc}{a} & -b \\ 0 & a \end{bmatrix}}$$ by the special case.

Therefore multiplying $S^{-1} * \begin{bmatrix} 1 & 0 \\ -\frac{c}{a} & 1 \end{bmatrix}$, we obtain A^{-1} as follows:

$$\frac{1}{ad-bc}\begin{bmatrix} \frac{ad-bc}{a} & -b \\ 0 & a \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\frac{c}{a} & 1 \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} \frac{ad}{a} & -b \\ -c & a \end{bmatrix}$$

or $$\boxed{A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$ exists if and only if $ad - bc \neq 0$.