2.3 Elimination and Permutation Matrices

Consider the row operation

$$B = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 2 & 2 & 2 & 4 \\ 3 & 5 & 4 & 8 \end{bmatrix} \xrightarrow[R2 - 2*R1]{R2 =} B' = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & -4 & -2 & -4 \\ 3 & 5 & 4 & 8 \end{bmatrix}$$

Now we define matrix multiplication by concatenation of our earlier definition: $\underset{m\times n}{E}\ \underset{n\times p}{B} = E\,[\vec{b}_1 \mid \vec{b}_2 \mid \cdots \mid \vec{b}_p]$
$:= [E\vec{b}_1 \mid E\vec{b}_2 \mid \cdots \mid E\vec{b}_p]^{m\times p}$.

Now consider $E^{3\times 3} = E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and put $B^{3\times 4} = \left[\begin{array}{c|c|c|c} 1 & 3 & 2 & 4 \\ 2 & 2 & 2 & 4 \\ 3 & 5 & 4 & 8 \end{array}\right]$, as above.

We claim $E_{21} B = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & -4 & -2 & -4 \\ 3 & 5 & 4 & 8 \end{bmatrix}$.

Indeed, consider $E_{21}\vec{b}$ for any column $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ of B.

The first entry of the product $E_{21}\vec{b}$ is unchanged because $(1,0,0)\cdot(b_1, b_2, b_3) = b_1$

Similarly the third entry is unchanged because

$$(0,0,1)\cdot(b_1, b_2, b_3) = b_3$$

But the second entry of the product becomes

$$(-2,1,0)\cdot(b_1\ b_2\ b_3) = b_2 - 2b_1.$$

Therefore since this is true for any column $\vec{b}$ of B, we obtain by the concatenation rule for the product $E_{21}B$ that $E_{21}B$ = matrix B with 2^{nd} row replaced by row2 − 2*row1

Therefore the row operation changing B into B' is effected by the matrix product: $\boxed{B' = E_{21}B}$

We use the notation E_{21} to denote that a multiple of row 1 is subtracted from row 2. The only non-diagonal entry of E_{21} is the 2,1 entry.

2.3.

Note also that $EB = \left[\begin{array}{c} (\text{row 1 of } E)^{1\times 3} B^{3\times 4} \\ \hline (\text{row 2 of } E)^{1\times 3} B^{3\times 4} \\ \hline (\text{row 3 of } E)^{1\times 3} B^{3\times 4} \end{array}\right]^{3\times 4}$,

where $[1\ 0\ 0]\begin{bmatrix} 1 & 3 & 2 & 4 \\ 2 & 2 & 2 & 4 \\ 3 & 5 & 4 & 8 \end{bmatrix} = [1\ 3\ 2\ 4]$

But $[-2, 1, 0]\begin{bmatrix} 1 & 3 & 2 & 4 \\ 2 & 2 & 2 & 4 \\ 3 & 5 & 4 & 8 \end{bmatrix} = [0\ -4\ -2\ -4]$.

<u>Example</u>. Suppose $B^{3\times 4}$ satisfies row1 + row2 = row3. Find a matrix $F^{3\times 3}$ such that $FB = \left[\begin{array}{c} \text{row 1 of } B \\ \hline \text{row 2 of } B \\ \hline 0\ 0\ 0\ 0 \end{array}\right]^{3\times 4}$.

By the formula for the product at the top of the page we only want to have off diagonal entries of F in the 3rd row of F. So, $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$ does the job since

$$FB = \begin{bmatrix} \text{row 1 of } B \\ \text{row 2 of } B \\ \text{row3} - \text{row2} - \text{row1} \end{bmatrix}.$$

<u>Permutation Matrix</u>. If we switch rows 1 and 3 of the identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and call the result $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, then: $PB^{3\times n} = \begin{bmatrix} [\text{row 1 of } P]\, B \\ [\text{row 2 of } P]\, B \\ [\text{row 3 of } P]\, B \end{bmatrix}^{3\times n}$

So the product PB switches row 1 and row 3 of B because $[\text{row 3 of } P]\,B = \boxed{\text{row 1 of } B}$ and $\boxed{\text{row 1 of } P}\,B = \boxed{\text{row 3 of } B}$

Notes 2 (p.3)

2.3 Exercises.

2.3 # 1. $E_{21}^{3\times3}$ subtracts 5 times row 1 from row 2: $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

E_{32} subtracts -7 times row 2 from row 3 (that is adds 7 times row 2 to row 3): $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$

P exchanges rows 1 and 2 and then rows 2 and 3 so we obtain in the end $\begin{bmatrix} [\text{row } 2] \\ [\text{row } 3] \\ [\text{row } 1] \end{bmatrix}$: $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

2.3 #2

$$E_{32} E_{21} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = E_{32} \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -35 \end{bmatrix}$$

$$E_{21} E_{32} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = E_{21} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}.$$

Note $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = P_{23} P_{12}$.

But $P \neq P_{12} P_{23} = \begin{bmatrix} [\text{row 3 of } I] \\ [\text{row 1 of } I] \\ [\text{row 2 of } I] \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

2.3 #16 (a) $x = 2y$ and $x + y = 33$

$$\begin{aligned} x - 2y &= 0 \\ x + y &= 33 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & 1 & 33 \end{array}\right] \xrightarrow[R2 = R2 - R1]{} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 3 & 33 \end{array}\right]$$

$3y = 33 \Rightarrow y = 11$

$x = 2y \Rightarrow x = 22$

(b) $5 = m.2 + c$
$7 = m.3 + c$

$$\begin{cases} c + 2m = 5 \\ c + 3m = 7 \end{cases} \qquad \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 1 & 3 & 7 \end{array}\right]$$

$$\xrightarrow[R2 = R2 - R1]{} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 2 \end{array}\right] \Rightarrow m = 2, \quad c = 5 - 2m = 5 - 4 = 1.$$

$$\boxed{m = 2, \quad c = 1}.$$

Notes 2 (p. 4)

~~2.3 Exercises~~

2.3 #29. $$E^{4\times 4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$E \cdot B = B'$ where $B' = \begin{bmatrix} [\text{row } 1 \text{ of } B] \\ [(\text{row } 2 - \text{row } 1) \text{ of } B] \\ [(\text{row } 3 - \text{row } 2) \text{ of } B] \\ [(\text{row } 4 - \text{row } 3) \text{ of } B] \end{bmatrix}$

So $$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

2.4 Matrix Operations and Block Multiplication

Basic definition of Matrix Multiplication: Suppose $A^{m\times n}$ and $B^{n\times p}$ are matrices. Write

$$A = \begin{bmatrix} \text{row } 1 \\ \text{row } 1 \\ \vdots \\ \text{row } m \end{bmatrix}^{m\times n} \quad \text{and} \quad B = \left[\text{col } 1 \,|\, \text{col } 2 \,|\, \ldots \,|\, \text{col } p\right]^{n\times p}$$

Here then each row in A is a horizontal n-vector (row vector)$^{1\times n}$ and each column of B is a vertical n-vector (column vector)$^{n\times 1}$. The product AB is defined as the $m\times p$ matrix C with i^{th} row and j^{th} column element c_{ij} of C given by

$$\boxed{c_{ij} = \begin{pmatrix} \text{row } i \\ \text{of } A \end{pmatrix} \cdot \begin{pmatrix} \text{col } j \\ \text{of } B \end{pmatrix}} \quad [= \text{dot product of } n\text{-vectors}]$$

Example 1. $$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}\begin{pmatrix} \alpha & \delta \\ \beta & \varepsilon \\ \gamma & \zeta \end{pmatrix} = \begin{pmatrix} a\alpha + b\beta + c\gamma & a\delta + b\varepsilon + c\zeta \\ d\alpha + e\beta + f\gamma & d\delta + e\varepsilon + f\zeta \end{pmatrix}^{2\times 2}$$

2.4 Matrix Operations

Example 2. $\begin{pmatrix} a \\ b \\ c \end{pmatrix}^{3\times 1} (\alpha \ \beta \ \gamma)^{1\times 3} = \begin{pmatrix} a\alpha & a\beta & a\gamma \\ b\alpha & b\beta & b\gamma \\ c\alpha & c\beta & c\gamma \end{pmatrix}^{3\times 3}$

[each dot product in the definition (row i) • (col j) is a product of reals.]

Block Multiplication Another way to define matrix multiplication is to first define

$$\boxed{[a_1, \ldots, a_n]^{1\times n} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}^{n\times 1} \overset{(*)}{=} [a_1 b_1 + \ldots + a_n b_n]^{1\times 1}}$$

(= one row on the left times one column on the right where row and column have equal numbers of entries)

We also define

$$\boxed{\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}^{m\times 1} (b_1, \ldots, b_p)^{1\times p} \overset{(**)}{=} \begin{pmatrix} a_1 b_1 & a_1 b_2 & \cdots & a_1 b_p \\ \vdots & & & \vdots \\ a_m b_1 & & \cdots & a_m b_p \end{pmatrix}^{m\times p}}$$

Now suppose we write any matrix $A^{m\times n} = [\text{col} 1 | \cdots | \text{col} n]$ and $B^{n\times p} = \begin{bmatrix} \text{row} 1 \\ \hline \text{row} 2 \\ \hline \vdots \\ \hline \text{row} n \end{bmatrix}$, then we can write:

$$A = [\vec{a}_1^{\,m\times 1} | \cdots | \vec{a}_n^{\,m\times 1}]$$

and $B = \begin{bmatrix} (\vec{r}_1{}^T)^{1\times p} \\ \hline \vec{r}_2{}^T \\ \vdots \\ \hline (\vec{r}_n{}^T)^{1\times p} \end{bmatrix}$, [where T denotes transpose and $r_1, r_2, \ldots$ are row vectors.]

Define now $\boxed{AB = (\vec{a}_1)^{m\times 1}(\vec{r}_1{}^T)^{1\times p} + \vec{a}_2 \vec{r}_2{}^T + \ldots + \vec{a}_n \vec{r}_n{}^T}$ $(***)$

This is a Block Multiplication defined (formally) by $(*)$, where each product in $(***)$ is in turn defined by $(**)$!