Math 313 Linear Algebra

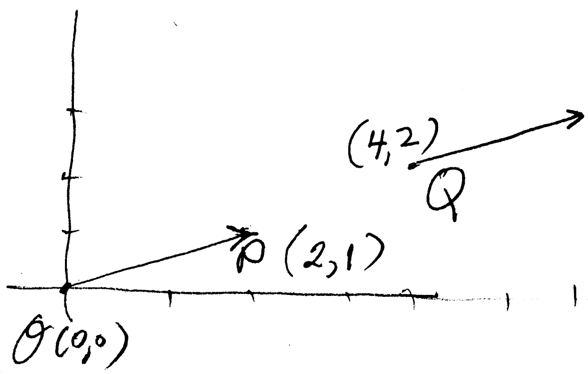
$\boxed{\text{Notes 1}}$ $\boxed{\text{Sects. 1.1, 1.2, 2.1, 2.2}}$

1.1 Algebra and Geometry of Vectors

Definition. A vector in $\mathbb{R}^2$ is simply a column of real numbers $\begin{bmatrix} a \\ b \end{bmatrix}$. (Sometimes we write it as a row for convenience)

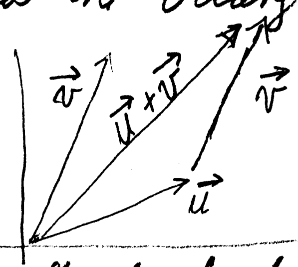
Geometrically this vector represents the displacement from the origin point $O(0,0)$ to the point $P(a,b)$; so $\begin{bmatrix} a \\ b \end{bmatrix} = \overrightarrow{OP}$ (displacement from O to P).

The displacement from $Q(x_0, y_0)$ to $R(x_1, y_1)$ is the vector $\overrightarrow{QR} = \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix}$. It may be represented geometrically also as the displacement from $O(0,0)$ to $P(x_1-x_0, y_1-y_0)$:



As vectors, $\overrightarrow{OP} = \overrightarrow{QR} = \begin{bmatrix} 6-4 \\ 3-2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Addition of vectors: Algebraically we easily add two vectors: $\vec{u} + \vec{v} = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \end{bmatrix}$. Geometrically, this rule becomes the triangle or parallelogram law:

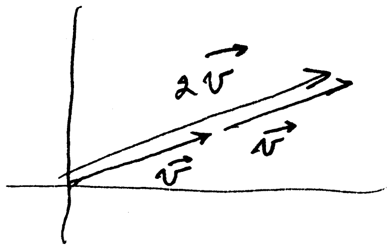


First we show both $\vec{u}$ and $\vec{v}$ in standard position with tails at the origin. Then we use parallel translation to put the tail of $\vec{v}$ at the head of $\vec{u}$ and finally connect the tail of $\vec{u}$ to the head of the translated $\vec{v}$. See also text, p. 3.

1.1. Scalar multiplication.

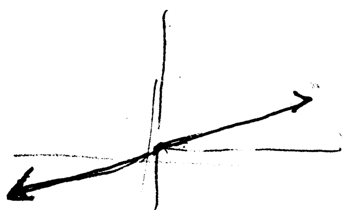
$2\vec{v} = 2\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} := \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix}$. This operation makes a vector twice as long in the same direction as $\vec{v}$.

Note that $2\vec{v} = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix} = \begin{bmatrix} v_1 + v_1 \\ v_2 + v_2 \end{bmatrix} = \vec{v} + \vec{v}$.



In general $c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$ for any real (scalar) c.

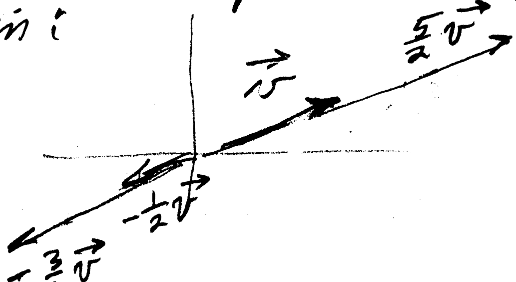
Note that scalar multiplication by $c = -1$ changes the direction of $\vec{v}$: $-1\vec{v} = (-1)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}$



$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$-\vec{v} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

$\vec{v}$ and $-\vec{v}$ lie on the same line through the origin (when placed in standard position). The set of all scalar multiples $\{c\vec{v} : c \in \mathbb{R}\}$ makes a whole collection of vectors in standard position whose heads make all points along the line of $\vec{v}$ through the origin:



We <u>identify</u> $\{c\vec{v} : c \in \mathbb{R}\}$ <u>as the line through $\mathcal{O}$ parallel to $\vec{v}$</u>.

Linear combinations.

Algebraically we now have $c\vec{v} + d\vec{w} = c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + d\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix} + \begin{bmatrix} dw_1 \\ dw_2 \end{bmatrix} = \begin{bmatrix} cv_1 + dw_1 \\ cv_2 + dw_2 \end{bmatrix}$. This is called a <u>linear combination</u> of the vectors $\vec{v}$ and $\vec{w}$. Here c and d are scalars.

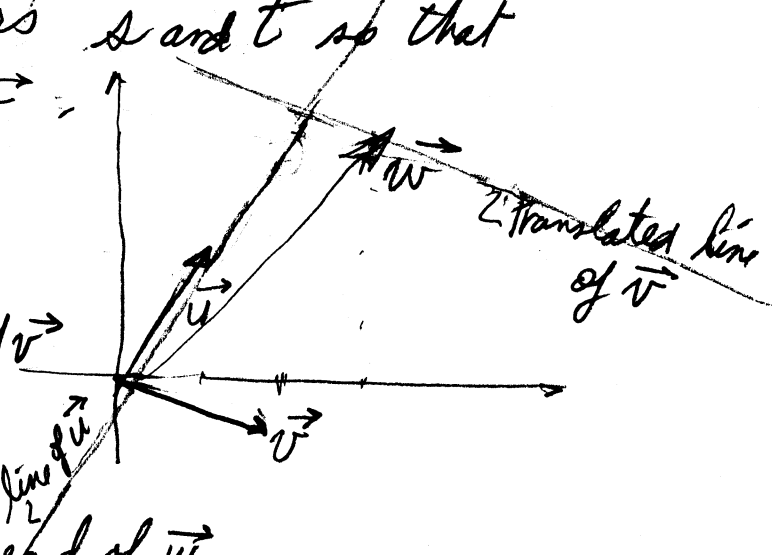
Notes 1 (p.3)

1.1. Example. Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Put $\vec{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Find scalars s and t so that

$$s\vec{u} + t\vec{v} = \vec{w}.$$

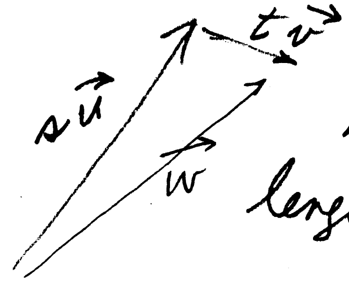
(a) Geometric solution:



(i) Draw a line through $\vec{u}$

(ii) Parallel translate the line of $\vec{v}$ along the line of $\vec{u}$ until the line of the translated $\vec{v}$ goes through the head of $\vec{w}$.

Now mark off a multiple $s\vec{u}$ of $\vec{u}$ along the line of $\vec{u}$ and also a multiple $t\vec{v}$ along the (translated) line of $\vec{v}$



In the picture both $s > 0$ and $t > 0$. s is calculated as the ratio of the length of $s\vec{u}$ to the length of $\vec{u}$, etc.

(b) Algebraic solution. Write $s\vec{u} + t\vec{v} = s\begin{bmatrix} 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} s+2t \\ 2s-t \end{bmatrix}$. Solve $\begin{bmatrix} s+2t \\ 2s-t \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ by $\begin{cases} s+2t = 3 & (1) \\ 2s - t = 4 & (2) \end{cases}$

Subtract 2 times equation (1) from equation (2)

Obtain $\begin{cases} s + 2t = 3 & (1) \\ 0 - 5t = 4 - 6 & (2)' \end{cases}$ (2)′ gives $t = \frac{-2}{-5} = 2/5 = 0.4$.

Now substitute $t = 2/5$ into (1) to obtain

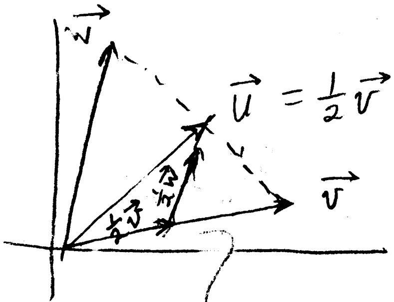
$$s + 4/5 = 3, \text{ or } s = \frac{15}{5} - \frac{4}{5} = 11/5 = 2.2.$$

The algebraic solution uses (algebraic) calculation in place of a ruler!

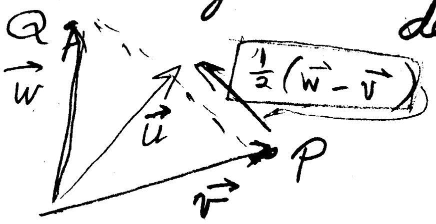
Notes 1, p.4

1.1 Exercises

1.1 #15.



In standard position the linear combination $\frac{1}{2}\vec{v} + \frac{1}{2}\vec{w}$ lies with its head at the midpoint of the segment joining $\vec{v} + \vec{w}$. One way to see this algebraically (it is plausible geometrically as shown) is to observe that $\frac{1}{2}\vec{v} + \frac{1}{2}\vec{w} = \vec{v} + \frac{1}{2}\vec{w} - \frac{1}{2}\vec{v}$
$= \vec{v} + \frac{1}{2}(\vec{w} - \vec{v})$. Here $\vec{w} - \vec{v}$ is the vector from the head of $\vec{v}$ to the head of $\vec{w}$. Indeed, if we denote these vector heads by points P and Q, then

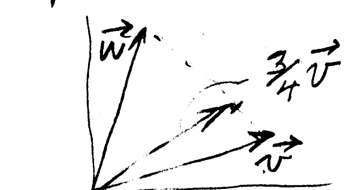


we have: $\vec{v} + \overrightarrow{PQ} = \vec{w}$; but this means $\overrightarrow{PQ} = \vec{w} - \vec{v}$ (by subtraction).

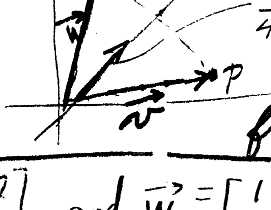
So $\vec{u} = \vec{v} + \frac{1}{2}$(displacement from P to Q) = midpoint of $\overline{PQ}$.

Similarly $\frac{3}{4}\vec{v} + \frac{1}{4}\vec{w} = \vec{v} + \frac{1}{4}(\vec{w} - \vec{v})$

= vector in standard position with head $\frac{1}{4}$ of the way from P to Q along $\overline{PQ}$:



Finally, $\frac{1}{4}\vec{v} + \frac{1}{4}\vec{w} = \frac{1}{2}\left(\frac{1}{2}\vec{v} + \frac{1}{2}\vec{w}\right) = \frac{1}{2}\vec{u}$, for $\vec{u}$ given in the first part:



$\frac{1}{4}\vec{v} + \frac{1}{4}\vec{w}$ lies halfway to the midpoint of $\overline{PQ}$ from the origin

1.1 #29. Let $\vec{u} = \begin{bmatrix}1\\3\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}2\\7\end{bmatrix}$ and $\vec{w} = \begin{bmatrix}1\\5\end{bmatrix}$. Find two different triples of scalars r, s, t so that $r\vec{u} + s\vec{v} + t\vec{w} = \begin{bmatrix}0\\1\end{bmatrix}$.

$$\begin{bmatrix} r + 2s + t \\ 3r + 7s + 5t \end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} \Leftrightarrow \begin{cases} r + 2s + t = 0 \\ s + 2t = 1 \end{cases} \Leftrightarrow \begin{cases} r - 3t = -2 \\ s + 2t = 1 \end{cases}$$

general solution: $t = t$, $s = 1 - 2t$, $r = -2 + 3t$

$r, s, t = (1, -1, 1)$ or $r, s, t = (-2, 1, 0)$, etc.