A directed polymer approach to once-oriented first passage site percolation in high dimensions*

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Abstract

Let η be a real valued random variable with a logarithmic moment generating function $\lambda(\beta) := \ln \mathbf{E} \exp(-\beta\eta)$ for all $\beta \geq 0$. Consider a random environment $\{\eta(z), z \in \mathbb{Z}^{(d+1)}\}$, for $d \geq 3$, where the $\eta(z)$ are independent copies of η. Let μ_d be the negative of the point to line time constant of once-oriented first passage site percolation on $\mathbb{Z}^{(d+1)}$. Consider the directed polymer in $\mathbb{Z}^{(d+1)}$ at inverse temperature $\beta > 0$ with random partition function $Z_n(\beta) = \mathbb{E} \exp(-\beta \sum_{k=1}^{n} \eta((k, S_k))$, where $\mathbb{E}$ denotes expectation with respect to a simple random walk $\{S_k\}$ in $\mathbb{Z}^d$. Denote the free energy $f(\beta) := \lim_{n\to\infty}(1/n)\mathbf{E} \ln Z_n(\beta)$. We use the fact that $\lim_{\beta\to\infty} f(\beta)/\beta = \mu_d$ to give very simple proofs of some upper and lower bounds for μ_d. The method yields in particular an asymptotic evaluation of the Gaussian last passage time constant: $\mu_d = \sqrt{2\ln(2d)} + o(1)$, as $d \to \infty$. In the case of a Bernoulli environment we establish upper and lower bounds for the first passage time constant τ_p such that: $\tau_p \leq 1 - C_1\sqrt{p}$, as $p \to 0$, and $\tau_p \geq 1 - C_2/\ln(1/p)$, as $p \to 0$, for constants C_1 and C_2 depending on d.

1 Introduction

Consider the time constant $\tau_p = \tau_{p,d}$ of a Bernoulli once-oriented first passage site percolation in $d+1$ dimensions with parameter p. One can relate τ_p to the free energy of a directed polymer model in a random environment (the Bernoulli field) at low temperature (inverse temperature β near infinity). This approach was already recognized by Comets and Yoshida [5], Sect. 7. Here we emphasize the fact that upper and lower bounds for τ_p as well as for the corresponding point to line first passage time constant in a general i.i.d. random environment (for example, a Gaussian field) may be obtained from the theory of the directed polymer. For one of our bounds we summarize a simple linear estimate on the free energy of the directed polymer model in a general random environment (Proposition 1, below) that is valid for all $d \geq 1$. Using this bound, that is closely related in spirit to Talagrand's [16] Proposition 1.1.3 (see also [2], Proposition 1.4), we obtain an asymptotic evaluation of the last passage time constant for the Gaussian field as $d \to \infty$. This

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result is extended in an Appendix for last passage time constants of certain environments whose underlying distributions are mixed with a Gaussian distribution. By using instead a slightly more sophisticated approach from the theory of the directed polymer (Lemma 1, below), for all $d \geq 3$ we obtain an upper bound for τ_p as follows: $\tau_p \leq 1 - C_1\sqrt{p}$, as $p \to 0$, for a constant C_1 depending on d. Proposition 1 and Lemma 1 together yield a lower bound: for all $d \geq 3$, $\tau_p \geq 1 - C_2/\ln(1/p)$, as $p \to 0$, for a constant C_2 depending on d.

We first introduce the once-oriented first passage site percolation model. Define the cone of sites $\mathbb{K} = \mathbb{K}_{d+1} := \{z = (t, y) \in \mathbb{Z}_+ \times \mathbb{Z}^d : t \geq 0 \text{ and } |y|_1 \leq t\}$, for $y = (y_1, \ldots, y_d) \in \mathbb{Z}^d$ and $|y|_1 := |y_1| + \cdots + |y_d|$. For a non-negative integer n let $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ be a sequence of nearest neighbor positions in $\mathbb{Z}^d$, with $\gamma_0 = \mathbf{0} \in \mathbb{Z}^d$ and such that for each $t = 0, 1, \ldots, n-1$, the increment $\gamma_{t+1} - \gamma_t = \pm e_i$ for some i^{th} unit coordinate vector e_i in $\mathbb{Z}^d$. The sequence of sites (t, γ_t), $t = 0, 1, \ldots, n$, describes a once-oriented path $\overrightarrow{\gamma_n} = \overrightarrow{\gamma_n}(\gamma)$ of length n in the cone $\mathbb{K}$. When $d = 1$ this model is equivalent to the usual directed site percolation on $\mathbb{Z}_+^2$. Here the usual directed site percolation model on $\mathbb{Z}_+^{d+1}$ assumes that exactly one coordinate of a path increases by 1 at each step instead of our assumption that it is always the first coordinate of $\overrightarrow{\gamma_n}$ that must increase. Thus for oriented paths in $d+1 = 3$ and higher space dimensions the two models are not equivalent because in the once-oriented case there are $2d$ choices for each step of the path compared to $d+1$ choices in the usual directed case. Our motivation for the once-oriented case is in fact its relation to the usual directed polymer model that we introduce below. Now introduce an i.i.d. field of Bernoulli random variables $\{\epsilon(z)\}$ over $z \in \mathbb{K}$, where for each z, $\mathbf{P}(\epsilon(z) = 0) = p$ and $\mathbf{P}(\epsilon(z) = 1) = 1 - p$ for a parameter $0 < p < 1$, and where we denote the probability and expectation relative to this field by $\mathbf{P}$ and $\mathbf{E}$ respectively. Denote the passage time over $\overrightarrow{\gamma_n}$ by

$$T(\overrightarrow{\gamma_n}) := \sum_{t=1}^{n} \epsilon((t, \gamma_t)),$$

and define the first passage time to pass from the origin at the vertex of $\mathbb{K}$ to $(n, \mathbf{0}) \in \mathbb{K}$ by $T_{0,n} := \min T(\overrightarrow{\gamma_n})$, where the minimum is extended over all once-oriented paths of length n with $\gamma_0 = \gamma_n = \mathbf{0} \in \mathbb{Z}^d$. We define for any $0 \leq m < n$ a once-oriented path $(m, \gamma_0), (m+1, \gamma_1), \ldots, (n, \gamma_{n-m})$ in $\mathbb{K}$, with $(\gamma_0, \gamma_1, \ldots)$ as above, and define

$$T_{m,n} := \min_{\gamma_0 = \gamma_{n-m} = \mathbf{0}} \sum_{t=1}^{n-m} \epsilon((m+t, \gamma_t)$$

Then obviously the collection of times $\{T_{m,n}\}$ is subadditive. By subadditivity we have that there exists a time constant $\tau_p = \tau_{p,d}$ such that

$$\lim_{n\to\infty} \mathbf{E}T_{0,n}/n = \tau_p = \inf_n \mathbf{E}T_{0,n}/n,$$

and, by Kingman's subadditive ergodic theorem, $\lim_{n\to\infty} T_{0,n}/n = \tau_p$, $\mathbf{P}$-a.s. and in $\mathbf{L}^1$.

Finally, denote by ξ_n^0 the set of "heights" $y \in \mathbb{Z}^d$ such that there is a once-oriented path $\overrightarrow{\gamma_n}$ starting from the origin of $\mathbb{K}$ and ending at (n, y) such that $T(\overrightarrow{\gamma_n}) = 0$. Define the once-oriented critical site percolation parameter

$$\overrightarrow{p_c}(\mathbb{K}_{d+1}) := \inf\{p : \mathbf{P}(\xi_n^0 \neq \emptyset \text{ for all } n) > 0\}.$$

Automatically $\tau_p = 0$ for $p > \overrightarrow{p_c}(\mathbb{K}_{d+1})$. It is also known by [10], [13] that $\tau_p > 0$ if and only if $p < \overrightarrow{p_c}(\mathbb{K}_{d+1})$. Furthermore, let N_n be the number of paths $\overrightarrow{\gamma_n}$ such that $T(\overrightarrow{\gamma_n}) = 0$. Then $N_n/(2dp)^n$ is a martingale relative to the fields $\sigma\{\epsilon(m,y) : m \leq n, (m,y) \in \mathbb{K}\}$, $n \geq 1$. Denote $\tilde{\rho}_d = \mathbb{P}(S_n = S'_n \text{ for some } n \geq 1)$, where S_n and S'_n are independent simple random walks in $\mathbb{Z}^d$. Denote also $\kappa = \inf\{k \geq 1 : S_k = S'_k\}$. By the method of Cox and Durrett [8], Sect. 2, we have the following estimate for the critical probability:

$$\begin{aligned} &1/(2d) \leq \overrightarrow{p_c}(\mathbb{K}_{d+1}) \leq \tilde{\rho}_d, \quad \text{where, } \tilde{\rho}_d = \mathbb{P}(\kappa = 1) + \mathbb{P}(\kappa = 2) + \cdots \\ &= 1/(2d) + (2d-1)/(2d)^3 + \cdots = 1/(2d) + 1/(4d^2) + O(1/d^3), \text{ as } d \to \infty. \end{aligned} \tag{1.1}$$

We also introduce the "origin to line" first passage times

$$\widehat{H}_{m,n} := \min_{\gamma_0 = \mathbf{0}} \sum_{t=1}^{n-m} \epsilon((m+t, \gamma_t)), \tag{1.2}$$

where the nearest neighbor sequence of positions $(\gamma_0, \gamma_1, \dots)$ in $\mathbb{Z}^d$ is as before, but there is no condition on the height γ_{n-m}. Even though the collection $\{\widehat{H}_{m,n}\}$ is not subadditive, J.B. Martin [12] gives a shape theorem as follows. Note that Martin studies the usual directed percolation on $\mathbb{Z}_+^{d+1}$; the proofs for the once-oriented case are obtained in the same way by the concentration inequality Lemma 3.1 of [12]. We need to check one condition, namely nondegeneracy of the shape, that is the analogue of the condition (ii) of Martin's [12], Theorem 5.1. First define the continuous analogue V of $\mathbb{K}$: $V := \{(t,y) \in \mathbb{R}_+ \times \mathbb{R}^d : |y|_1 \leq t\}$. For $z = (n,y) \in \mathbb{K}$, define $T(z) := \inf_{\gamma_0 = \mathbf{0}, \gamma_n = y} T(\overrightarrow{\gamma_n})$ and for $x \in V$, put $h(x) := \inf_n \mathbf{E}T(\lfloor nx \rfloor)/n$. Finally define $C := \{x \in V : h(x) \leq 1\}$ and $C_t := \{x \in V : T(x) \leq t\}$. By subadditivity, h is subadditive and C is convex. Further since $h(t,y) = h(t,-y)$, it follows by subadditivity of h that $\inf_{|y|_1 \leq 1} h(1,y) \geq h(1,\mathbf{0}) = \tau_p$. Thus the desired nondegeneracy of shape condition, that is for some ball B centered at the origin we have that $B \cap V \subset C$ holds, is satisfied when $\tau_p > 0$. Hence under this condition, we have that for any $0 < \delta < 1$, with $\mathbf{P}$-probability 1,

$$(1-\delta)C \subset C_t/t \subset (1+\delta)C \text{ for all large } t.$$

It follows by convexity of C and symmetry along the axis of C as in [7] or [10] that $\lim_{n\to\infty} \widehat{H}_{0,n}/n = \tau_p$, $\mathbf{P}$-a.s.; obviously this last statement is true even if $\tau_p = 0$ since $0 \leq \widehat{H}_{0,n} \leq T_{0,n}$. Furthermore, since the sequence of expected values $\mathbf{E}\widehat{H}_{0,n}$ *is* subadditive we may therefore also write (using the previous a.s. limit and the subadditive lemma) that

$$\tau_p = \inf_n \mathbf{E}\widehat{H}_{0,n}/n. \tag{1.3}$$

We now introduce the directed polymer model; see [4] for a physical motivation. Define the (random) partition function Z_n at level n as a function of the inverse temperature $\beta > 0$ by

$$Z_n(\beta) := (2d)^{-n} \sum_{\gamma} \exp(-\beta T(\overrightarrow{\gamma_n})),$$

where the sum is over all $(2d)^n$ nearest neighbor sequences $(\gamma_0, \gamma_1, \dots, \gamma_n)$ in $\mathbb{Z}^d$ starting from the origin as above. An alternative representation of the partition function is obviously given by

$Z_n(\beta) = \mathbb{E}\exp(-\beta\sum_{k=1}^n \epsilon(k, S_k))$, where $\{S_k\}$ is a simple symmetric random walk in $\mathbb{Z}^d$ and where $\mathbb{E}$ denotes the expectation with respect to this random walk. Thus $Z_n(\beta)$ is the partition function of a directed polymer in a random environment; the environment is the Bernoulli field $\{\epsilon(z)\}$ and, in the formula $Z_n = \text{ave.}\exp[-\text{energy}]$, the energy of the path $\overrightarrow{\gamma_n}$ is simply: energy $= \beta T(\overrightarrow{\gamma_n})$. Finally we define the (random) Gibbs measure on the collection of nearest neighbor sequences $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ by

$$\mu_n(\gamma) := (2d)^{-n}\exp(-\beta T(\overrightarrow{\gamma_n}))/Z_n(\beta),$$

and write $\langle\cdot\rangle^{(n)}$ for the integration operation with respect to μ_n. When $\beta = 0$ we simply obtain uniform measure on γ (independent of the environment) that is of course the measure of the simple symmetric random walk. In a series of papers going back to [1], where Bolthausen gives a martingale proof of the original result of Imbrie and Spencer [9] on the diffusive behavior of the height γ_n for small $\beta > 0$ and high dimensions $(d+1 \geq 4)$, namely $\langle|\gamma_n|^2\rangle^{(n)}/n \to 1$ $\mathbf{P}$-a.s., a martingale theory has been established for the directed polymer in a random environment (see [4],[5]). Here in general one puts a random variable η in place of the Bernoulli variable ϵ such that $\lambda(\beta) := \ln \mathbf{E}\exp(-\beta\eta)$ exists for all $\beta \geq 0$. Obviously we have that $\mathbf{E}Z_n(\beta) = \exp(n\lambda(\beta))$, and the martingale theory leads to a dichotomy: for all $\beta \geq 0$, $\lim_{n\to\infty} Z_n(\beta)\exp(-n\lambda(\beta)) = \widetilde{Z}_\infty(\beta)$ exists, and either

$$\text{(i) } \widetilde{Z}_\infty(\beta) > 0 \;\; \mathbf{P}\text{-a.s. or, (ii) } \widetilde{Z}_\infty(\beta) = 0 \;\; \mathbf{P}\text{-a.s.}$$

In case (i) it is said that weak disorder holds, while under (ii) it is said that strong disorder holds. For example weak disorder holds if $d+1 \geq 4$ and $\beta > 0$ is small, and, further, if (ii) holds for some $\beta > 0$ then there is a critical inverse temperature β_c above which the strong disorder holds [5]. For $d = 1$, Comets and Yoshida [5] show that by contrast with the high dimensional case, $\beta_c = 0$, so that in the plane there is sufficient interaction among the oriented paths to yield the strong disorder no matter how high the temperature.

By writing $\ln Z_{n+m} = \ln Z_n + \ln\sum_y \mu_n(\gamma_n = y)Z^y_{n,m}$, one finds by Jensen's inequality and the fact that $\mathbf{E}\ln Z^y_{n,m} = \mathbf{E}\ln Z_m$, that $\mathbf{E}\ln Z_n$ is superadditive (see [3], Proposition 1.5). Hence by superadditivity the so-called free energy $f(\beta)$ exists as follows:

$$\lim_n \mathbf{E}(\ln Z_n)/n = f(\beta) = \sup_n \mathbf{E}(\ln Z_n)/n. \tag{1.4}$$

Jensen's inequality immediately gives that $f(\beta) \leq \lambda(\beta)$. By [3] one also obtains a concentration inequality: $\mathbf{P}(|(1/n)\ln Z_n - f(\beta)| > \delta) \leq \exp(-\delta^2 n/c)$, so that in particular by the Borel-Cantelli lemma,

$$\lim_{n\to\infty}(1/n)\ln Z_n = f(\beta) \;\; \mathbf{P}\text{-a.s.} \tag{1.5}$$

It is well known that $f(\beta)$ is convex in β (see below). Further it is shown by [5] using an FKG inequality that the so-called Lyapunov exponent $\psi(\beta) := \lambda(\beta) - f(\beta)$ is non-decreasing. Comets and Yoshida ([5], Theorem 3.2) use this monotonicity property together with the martingale theory developed in [4] to establish that a phase transition from weak disorder to strong disorder occurs at most once and that $\psi(\beta) = 0$ for $0 \leq \beta \leq \beta_c$. For $d \geq 2$ an open problem is whether there exists a second phase transition $\beta_c^\psi > \beta_c$ such that $\psi(\beta) = 0$ for $\beta < \beta_c^\psi$ and $\psi(\beta) > 0$ for $\beta > \beta_c^\psi$ (see [5], Remark 3.2). In $d = 1$ Comets and Vargas [6] have shown that there is no such second transition so that $\psi(\beta) > 0$ for all $\beta > 0$.

2 Bounds on the free energy

Our object in this Section is to obtain a bound on the free energy (Proposition 1, below) that we can then apply to the problem of estimating μ_d in general and τ_p in particular. It is still unknown whether τ_p is convex for $p \in [0, \overrightarrow{p_c}(\mathbb{K}_{d+1})]$ though there is some evidence against it in high dimensions (see [11]). By our approach we will be able to show that $\tau_{p,d}$ has infinite slope at $p = 0$ for all $d \geq 3$. To see how the free energy is related to the time constant, we first consider the double limit $\lim_{n,\beta\to\infty} \ln Z_n(\beta)/(n\beta)$ as in [5], Sect. 7. Let $K = K_n$ denote the number of oriented paths along which $\widehat{H}_{0,n}$ is attained. Then we have (for a general random environment) the simple estimate

$$\ln[\exp(-\beta\widehat{H}_{0,n})K/(2d)^n]/(n\beta) \leq \ln Z_n(\beta)/(n\beta) \leq -\widehat{H}_{0,n}/n.$$

Therefore, since $K \geq 1$, we have for all $n \geq 1$ and $\beta > 0$

$$-\widehat{H}_{0,n}/n - \ln(2d)/\beta \leq \ln Z_n(\beta)/(n\beta) \leq -\widehat{H}_{0,n}/n. \tag{2.1}$$

Obviously the relation (2.1) continues to hold when the expectation $\mathbf{E}$ is applied to both $\ln Z_n(\beta)$ and $\widehat{H}_{0,n}$ therein. Now denote by $\mu_d = \mu_d(\eta)$ the time constant associated to the superadditive sequence $-\mathbf{E}\widehat{H}_{0,n}$ with η in place of ϵ in the definition (1.2). So μ_d is the negative of the point to line first passage percolation time constant for the general random environment. In case η is symmetric this is of course the corresponding last passage time constant. Thus by the above observations we have by (1.4) that

$$\lim_{\beta\to\infty} f(\beta)/\beta = \lim_{n\to\infty} -\mathbf{E}\widehat{H}_{0,n}/n = \sup_{n\geq 1} -\mathbf{E}\widehat{H}_{0,n}/n = \mu_d. \tag{2.2}$$

In particular these two limits have the common value $\mu_d = -\tau_{p,d}$ in the Bernoulli case (as noted in [5]). In the Gaussian case ($\eta \sim \mathcal{N}(0,1)$) we have by [2] that for $\beta > 0$,

$$f(\beta) \leq \min(\beta^2/2, \beta\sqrt{2\ln(2d)}), \tag{2.3}$$

so in this case one immediately obtains by (2.2) that

$$\mu_d(\mathcal{N}(0,1)) \leq \sqrt{2\ln(2d)}. \tag{2.4}$$

Remark 1 *One can show directly (see the Appendix) that* $\mathbf{E}\max(g_1, \ldots, g_{2d}) = \sqrt{2\ln(2d)} + o(1)$ *as* $d \to \infty$*, where* $g_1, \ldots, g_{2d}$ *denote i.i.d.* $\mathcal{N}(0,1)$ *r.v.'s. So by* (2.2) *with* $n = 1$ *and* (2.3) *we obtain in the Gaussian case that likewise,* $\mu_d = \sqrt{2\ln(2d)} + o(1)$ *as* $d \to \infty$*. We extend this asymptotic evaluation of* μ_d *for a class of mixtures in Proposition 2, below.*

Note that (1.4) gives a general lower bound for the free energy and that (2.2) gives a lower bound for μ_d. For illustration in the Bernoulli case with $n = 1$ in (2.2) we obtain $\tau_{p,d} \leq (1-p)^{2d}$. One can of course attempt to apply (2.2) further with n larger than 1; however computational difficulties arise. We will calculate another upper bound for τ_p in Sect. 3, below. To find a lower bound for τ_p in particular, we summarize the following inequality for $f(\beta)$ that in application leads to an upper bound for μ_d in general (see Remark 2, below).

Proposition 1 *We have the following linear lower bound for the free energy:*

$$f(\beta) \geq \beta\mu_d - \ln(2d).$$

Proof. Write

$$\mathbf{E}(\ln Z_n) = \mathbf{E}\ln[(2d)^{-n}\sum_{\gamma}\exp(-\beta T(\overrightarrow{\gamma_n}))] \geq \mathbf{E}\max_{\gamma}(-\beta T(\overrightarrow{\gamma_n})) - n\ln(2d),$$

where we have estimated the sum from below by the largest summand. Now divide by n and hence obtain by the definition (1.2) that

$$\mathbf{E}(\ln Z_n)/n \geq -\beta\mathbf{E}\widehat{H}_{0,n}/n - \ln(2d).$$

We now take the limit as $n \to \infty$ and use (1.4) and (2.2) to evaluate the limits on the two sides of this last inequality.□

It is well known that $f_n(\beta) := \mathbf{E}\ln Z_n(\beta)$ is convex (since the second derivative in β is a variance with respect to Gibbs measure). In the Bernoulli case $f_n(\beta)$ is non-positive and decreasing for $\beta \geq 0$ (as seen by applying Jensen's inequality w.r.t $\mathbf{E}$) because these properties hold for $\lambda(\beta; p) := \ln(p + (1-p)\exp(-\beta))$, so taking limits in (1.4) we obtain these properties for $f(\beta)$. Note that in the Gaussian case instead $f(\beta)$ is non-negative and increasing for $\beta \geq 0$ since now these are the properties of $\lambda(\beta) = \beta^2/2$. Thus by (2.2), in these examples, Proposition 1 simply gives a bound on the constant C_d for the tangent line $\lambda = \mu_d\beta + C_d$ of $f(\beta)$ at $\beta = \infty$; namely, $C_d \geq -\ln(2d)$ in either case. Notice that in the Gaussian case we have that $\beta_c^\psi \leq \mu_d$ by $\lambda'(\beta) = \beta$ and convexity of $f(\beta)$.

Remark 2 *The inequality* (2.4) *may be obtained from the convexity of* $f(\beta)$ *and* $f(\beta) \leq \lambda(\beta) = \beta^2/2$ *using Proposition 1 as follows. The tangent line* $\lambda = \beta\mu_d - \mu_d^2/2$ *to* $\lambda(\beta)$ *at* $\beta = \mu_d$ *must lie on or above the tangent to* $f(\beta)$ *at* $\beta = \infty$*. Therefore by Proposition 1,* $-\mu_d^2/2 \geq -\ln(2d)$*. Therefore by Remark 1 the constant* $\ln(2d)$ *in Proposition 1 is asymptotically best possible as* $d \to \infty$ *in the Gaussian case. Further, since* $\tau_{p,d} = 0$ *at* $p = \overrightarrow{p_c}(\mathbb{K}_{d+1})$*, and since the value of* $-\lambda(\beta; p)$ *at* $\beta = \infty$ *is* $\ln(1/p)$*, we also see by* (1.1) *that the constant* $\ln(2d)$ *in Proposition 1 is asymptotically best possible in the Bernoulli case.*

3 Bounds on τ_p

In this Section we obtain upper and lower bounds on τ_p when $d \geq 3$. To obtain an upper bound on τ_p that will show an infinite slope at $p = 0$ we use the following result of Comets and Yoshida ([5], Theorem 3.2) coupled with the martingale proof of weak disorder in dimension $d \geq 3$ for a sufficiently small $\beta > 0$ given by Song and Zhou for a general random environment ([14]; see also Theorem 2.3.2 of [4]). To state this result we introduce the probability π_d of return to the origin of a simple random walk S_n in $\mathbb{Z}^d$ for $d \geq 3$: $\pi_d := \mathbb{P}(S_n = \mathbf{0}$ for some $n \geq 1)$. It is known that $\pi_3 = .3405\ldots$; see Spitzer ([15], p. 123).

Lemma 1 *Let $d \geq 3$ and let $\lambda(\beta) = \ln \mathbf{E} \exp(-\beta\eta)$ be the logarithmic moment generating function for a random environment generated by the r.v. η. If $\lambda(2\beta) - 2\lambda(\beta) \leq \ln(1/\pi_d)$, then the free energy of the directed polymer at inverse temperature $\beta > 0$ satisfies $f(\beta) = \lambda(\beta)$.*

Note that by Lemma 1, $\pi_d \geq \overrightarrow{p_c}(\mathbb{K}_{d+1}) \geq 1/(2d)$, as noted by [5], Sect. 7. Also one may show again by a similar approach as shown in Cox and Durrett [8], Sect. 2, that

$$\pi_d = 1/(2d) + (4d-1)/(2d)^3 + \cdots = 1/(2d) + 1/(2d^2) + O(1/d^3), \text{ as } d \to \infty. \tag{3.1}$$

We apply Lemma 1 in the Bernoulli case for $d \geq 3$ simply as follows. We first solve for $\beta = \beta_0$ such that $\lambda(2\beta) - 2\lambda(\beta) = \ln(1/\pi_d)$. Second, we argue by Lemma 1 and concavity of $-f(\beta)$ that, since $-\lambda(\beta) = -f(\beta)$ for $0 \leq \beta \leq \beta_0$, we have $-\lambda'(\beta_0) \geq \tau_p$. In the first step, we obtain, by substituting $\lambda(\beta) = \ln(p + (1-p)\exp(-\beta))$ and solving the resulting quadratic equation in $\exp(-\beta_0)$, that

$$\begin{aligned} &\exp(-\beta_0) \\ &= \left(-p(1-p) + \sqrt{\pi_d(1-\pi_d)(p-p^2)}\right) / ((1-\pi_d) - (2-\pi_d)p + p^2) \\ &= \left(\sqrt{\pi_d/(1-\pi_d)}\right)\sqrt{p} - p/(1-\pi_d) + O(p^{3/2}), \text{ as } p \to 0. \end{aligned} \tag{3.2}$$

In the second step, after substituting expression (3.2) into $-\lambda'(\beta_0)$, and after denoting $C_0 = \sqrt{\pi_d/(1-\pi_d)}$ and $C = 1/(1-\pi_d)$, we find

$$\begin{aligned} \tau_p &\leq \left(C_0\sqrt{p} - Cp + O(p^{3/2})\right) / \left(C_0\sqrt{p} - (C-1)p + O(p^{3/2})\right) \\ &= 1 - (1/C_0)\sqrt{p} + O(p), \text{ as } p \to 0. \end{aligned} \tag{3.3}$$

This establishes our upper bound for τ_p. To obtain a lower bound we apply Lemma 1 again at β_0, this time using in addition by Proposition 1 the bound $-\lambda(\beta_0) \leq \ln(2d) + \tau_p\beta_0$. Hence by (3.2) we obtain

$$\begin{aligned} \tau_p &\geq \left(-\ln(C_0\sqrt{p} + O(p)) - \ln(2d)\right) / -\ln(C_0\sqrt{p} + O(p)) \\ &\geq 1 - 2\ln(2d)/\ln(1/p) + O(1/\ln^2(p)). \end{aligned} \tag{3.4}$$

We conclude this section by illustrating our method once again in the exponential case. The novelty of our approach is its simplicity. Let η be a unit exponential random variable, so $\lambda(\beta) = -\ln(\beta+1)$. Let $\nu_d := -\mu_d$ be the corresponding point to line first passage time constant. By (2.2) with $n = 1$ we have the following upper bound: $\nu_d \leq \mathbf{E}\min(\eta_1, \ldots, \eta_{2d}) = 1/(2d)$. On the other hand, let β be chosen so that $-\lambda'(\beta) = \nu_d$, that is $\beta = 1/\nu_d - 1$. Then by Proposition 1 we have that $\beta\nu_d + \ln(2d) \geq -f(\beta) \geq -\lambda(\beta) = \ln(1/\nu_d)$. So $\ln(\nu_d) \geq \nu_d - 1 - \ln(2d)$. Hence $\nu_d \geq 1/(2de)$, all $d \geq 1$. Obviously one can iterate this last result and therefore obtain in case $d = 1$ the bounds $0.23196\cdots \leq \nu_1 \leq 1/2$. For high d we can improve the upper bound as follows. Solve $\lambda(2\beta_0) - 2\lambda(\beta_0) = \ln(1/\pi_d)$ by $\beta_0 = ((1-\pi_d)/\pi_d)(1 + 1/\sqrt{1-\pi_d})$. Hence, by (3.1) we obtain that $-\lambda'(\beta_0) = 1/(\beta_0+1) \sim 1/(4d)$ is an asymptotic upper bound for the first passage time constant ν_d as $d \to \infty$.

4 Appendix

Let $g_1, \ldots, g_N$ denote i.i.d. $\mathcal{N}(0,1)$ r.v.'s. We calculate a lower bound for $\mathbf{E}\max(g_1, \ldots, g_N)$ as follows. Let $\Phi(x)$ denote the standard normal cumulative distribution function. Thus we have that

$\mathbf{P}((\max_{1\le i\le N} g_i)\le x)=\Phi(x)^N$. Denote $I_N:=\mathbf{E}(\max_{1\le i\le N} g_i)=\int_{-\infty}^{\infty} x d(\Phi(x)^N)$. By a change of variables $u=\Phi(x)$ we write $I_N=\int_0^1 \Phi^{-1}(u)Nu^{N-1}du$. By [16], (A.37)-(A.38), we have the estimate:

$$1-\exp(-t^2/2)\le \Phi(t)\le 1-\exp(-t^2/2)/(L(1+t)),\ \text{all } t\ge 0, \tag{4.1}$$

for some absolute constant $L>0$. By the upper bound in (4.1) we obtain easily the following.
Claim: $\Phi^{-1}(u)\ge\sqrt{1-2\ln(1-u)-2\ln(L)}-1,\ \text{all } u\ge 1-1/L.$
Proof of claim. If $u\ge 1-1/L$ then $x=\Phi^{-1}(u)$ exceeds the value t such that $\exp(-t^2/2)/(L(1+t))=1-u$. By applying the inequality $1+t\le\exp(t)$, $t\ge 0$, we have that $\exp(-t^2/2)/(1+t)\ge\exp(-t^2/2-t)$. Therefore the value t' that satisfies $\exp(-t^2/2-t)/L=1-u$ satisfies $t'\le t\le x$. By solving this last equation for t', the claim is proved. □

Now write

$$\begin{aligned} I_N &= \left(\int_0^{1-1/L}+\int_{1-1/L}^1\right)\Phi^{-1}(u)Nu^{N-1}du \\ &\ge \int_{-\infty}^{\Phi(1-1/L)} x\Phi(x)^{N-1}d\Phi(x)+\int_{1-1/L}^1(\sqrt{1-2\ln(1-u)-2\ln(L)}-1)Nu^{N-1}du. \end{aligned} \tag{4.2}$$

The first term on the right side of (4.2) is bounded below by $N(1/2)^{N-1}\int_{-\infty}^0 xd\Phi(x)$, and so is negligible as $N\to\infty$. Denote the second integral on the right side of (4.2) by Q_N. Then make the substitution $u=1-w/N$ and so write

$$\begin{aligned} Q_N &= \int_0^{N/L}(\sqrt{1-2\ln(w/N)-2\ln(L)}-1)(1-w/N)^{N-1}dw \\ &\ge \int_{1/N}^{\ln\ln(N)}(\sqrt{1-2\ln(w/N)-2\ln(L)}-1)\exp(-w/(1-\ln\ln(N)/N))dw, \end{aligned} \tag{4.3}$$

where we have used at the last step the inequality $\ln(1-a)\ge -a/(1-a)$, $0\le a<1$, so that for $w\le\ln\ln(N)$ we have $(1-w/N)^N\ge\exp(-w/(1-\ln\ln(N)/N))$. Finally write $-2\ln(w/N)=2\ln(N)-2\ln(w)\ge 2\ln(N)-2\ln\ln\ln(N)$ for $w\le\ln\ln(N)$. Hence we obtain

$$Q_N\ge\sqrt{2\ln(N)}(1-C\ln\ln\ln(N)/\ln(N)), \tag{4.4}$$

for a constant $C>0$. We conclude that $I_N\ge Q_N+o(1)\ge\sqrt{2\ln(N)}+o(1)$ as $N\to\infty$.

We extend the result of the last paragraph as follows. Let $G(x):=\Phi(x/\sigma)$ be the distribution function of $N(0,\sigma^2)$, and let F be a distribution function of a symmetric random variable η_1. Denote $\lambda_1(\beta)=\ln\mathbf{E}(\exp(-\beta\eta_1))$. Assume that

$$\text{(A)}\qquad \limsup_{\beta\to\infty}(1/\beta^2)\lambda_1(\beta)<\sigma^2/2.$$

Let $H(x):=(1-\theta)G(x)+\theta F(x)$ be a mixture of F and G for some $0<\theta<1$. Then we have the following.

Proposition 2 *Under assumption (A) the time constant $\mu_d(H)$ associated to the time distribution H satisfies $\mu_d(H)=\sigma\sqrt{2\ln(2d)}+o(1)$, as $d\to\infty$.*

To prove this result, first find $\sigma_1^2<\sigma^2$ and β_1 so large that for $\beta\ge\beta_1$ we have $\lambda_1(\beta)\le\sigma_1^2\beta^2/2$. We first show that

$$\mu_d(H)\ge\sigma\sqrt{2\ln(2d)}+o(1),\ \text{as } d\to\infty. \tag{4.5}$$

To do this note that by symmetry of η_1, Markov's inequality and assumption (A), that for all $\beta \geq \beta_1$ we have

$$\mathbf{P}(\eta_1 \geq t) \leq \exp(-\beta t)\mathbf{E}\exp(\beta\eta_1) \leq \exp(-\beta t + \beta^2\sigma_1^2/2). \tag{4.6}$$

Now put $t_1 := \sigma_1^2\beta_1$. Then by (4.6) for any $t \geq t_1$ we can choose $\beta := t/\sigma_1^2$ and thus obtain

$$\mathbf{P}(\eta_1 \geq t) \leq \exp(-t^2/(2\sigma_1^2)), \text{ all } t \geq t_1. \tag{4.7}$$

By (4.7) and (4.1) it now follows that for $t \geq t_1$,

$$1 - H(t) \leq (1-\theta)\exp(-t^2/(2\sigma_1^2)) + \theta\exp(-t^2/(2\sigma^2))/(L(1+t/\sigma)). \tag{4.8}$$

Hence we can follow the proof above to obtain that $H^{-1}(u) \geq \sigma\sqrt{1 - 2\ln(1-u) + 2\ln(L/\theta)} - 1$, for all $u \geq 1 - \theta/L$, and so obtain our desired lower bound (4.5).

For an upper bound on $\mu_d(H)$ we apply Proposition 1. First note that the logarithmic moment generating function λ_H of H is written $\lambda_H(\beta) = \ln((1-\theta)\exp(\lambda_1(\beta)) + \theta\exp(\beta^2\sigma^2/2))$. Let $\epsilon > 0$ be chosen suitably small such that $\delta := \sigma^2 - (1+\epsilon)\sigma_1^2 > 0$. By the Hölder's inequality and assumption (A), we estimate that for all $\beta \geq \beta_1$,

$$\begin{aligned} \mathbf{E}(\eta_1\exp(-\beta\eta_1)) &\leq \left(\mathbf{E}|\eta_1|^{(1+\epsilon)/\epsilon}\right)^{\epsilon/(1+\epsilon)} \left(\mathbf{E}\exp(-(1+\epsilon)\beta\eta_1)\right)^{1/(1+\epsilon)} \\ &\leq C_\epsilon \exp(\beta^2(\sigma^2-\delta)/2). \end{aligned} \tag{4.9}$$

In view of (4.9) it is an easy matter to show that $\lambda_H'(\beta) = \beta\sigma^2 + O(\beta\exp(-\beta^2\delta/2))$, as $\beta \to \infty$. Hence because we already have that (4.5) holds, we easily determine that $\lambda_H'(\beta) = \mu_d(H)$ will hold for $\beta = \mu_d(H)/\sigma^2 + o(1)$ as $d \to \infty$. Hence by Proposition 1 applied for this value of β we obtain for $\mu_d = \mu_d(H)$ that

$$(\sigma^2/2)\left(\mu_d/\sigma^2 + o(1)\right)^2 \geq \left(\mu_d/\sigma^2 + o(1)\right)\mu_d - \ln(2d), \tag{4.10}$$

where $o(1) = O(d^{-\delta'})$ as $d \to \infty$ for some $\delta' > 0$. Combining (4.5) and (4.10) we have therefore proved Proposition 2. Note finally that Proposition 2 continues to hold if we choose to let θ depend on d such that $\theta >> 1/d^\epsilon$ for every $\epsilon > 0$.

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