Math 132 Spring 2008 Abrams **Exam 3 - SOLUTIONS**

Show all your work. No credit will be given for answers which are not accompanied by supporting computations. Circle your answer when appropriate. Use the back of the sheet if necessary. As usual, notation counts! Be clear and precise with your answers, justifying responses appropriately. Make sure to include the units in your answer when warranted. You may use a standard calculator, one which does not have graphing capability or a QWERTY keyboard. Good luck!

1. (2 pt) Complete PRECISELY this definition of the definite integral. If f is a function defined on $[a,b]$, the definite integral from a to b is the number

$$\int_a^b f(x)dx = LIM_{\max \Delta x_i \to 0} \sum_{i=1}^n f(x_i^*)\Delta x_i$$

(Your answer should be some sort of limit.)

2. (2 pt) PRECISELY complete this statement of the Evaluation Theorem: If f is a continuous function on the interval $[a,b]$, then

$\int_a^b f(x)dx = F(b) - F(a)$ where $F(x)$ is any antiderivative for $f(x)$.

3. (5 pt) Draw the function $f(x) = 3\sin x$ on the interval $[0,\pi]$. Then shade the region which lies under the function, and over the x–axis, between $x = 0$ and $x = \pi$. Then find the area of this region.

Since $f(x) \geq 0$ on the interval $[0,\pi]$, the area is the definite integral $\int_0^\pi 3\sin x dx$. But $F(x) = -3\cos x$ is an antiderivative for $f(x)$, so by the Evaluation Theorem we have that the area is $F(\pi) - F(0) = -3\cos\pi - (-3\cos 0) = -3 \cdot (-1) - (-3 \cdot 1) = 6$ square units.

4. (4 pt each) Evaluate these definite integrals.

a. $\int_{-2}^0 (z^2 - z^5)dz$. Use the Evaluation Theorem (it's OK to use since $z^2 - z^5$ is continuous on $[-2,0]$.) $F(z) = \frac{1}{3}z^3 - \frac{1}{6}z^6$ is an antiderivative for $z^2 - z^5$, so

$$\int_{-2}^0 (z^2 - z^5)dz = F(0) - F(-2) = (0-0) - (\tfrac{1}{3}(-2)^3 - \tfrac{1}{6}(-2)^6) = -(\frac{-8}{3} - \frac{64}{6}) = -(-\frac{80}{6}) = \frac{40}{3}.$$

b. $\int_0^3 (e^x + x^2)dx$. Use the Evaluation Theorem (it's OK to use since $e^x + x^2$ is continuous on $[0,3]$.) $F(x) = e^x + \frac{1}{3}x^3$ is an antiderivative for $e^x + x^2$, so

$$\int_0^3 (e^x + x^2)dx = F(3) - F(0) = (e^3 + \tfrac{1}{3}3^3) - (e^0 + \tfrac{1}{3}0^3) = (e^3 + 9) - (1) = e^3 + 8 \approx 28.086.$$

5. (2 pt) PRECISELY complete this statement of the Net Change Theorem, using integrals and derivatives. Let F be any function for which $F'(x)$ is continuous. Then the integral of a rate of change is the net change. In mathematical symbols,

$$\int_a^b F'(x)dx = F(b) - F(a)$$

6. (2 pt) If oil leaks from a tank at a rate of $r(t)$ gallons per minute at time t, what does $\int_0^{180} r(t)dt$ represent?

The total amount of oil (in gallons) which leaks out of the tank during the time period between time 0 and time 180 (minutes). In other words, it's the number of gallons of oil which leaks out in the first three hours.

7. (6 pt total) The velocity of an object is given by $v(t) = 30 - 15t$ on the time interval $[0,3]$, where $v(t)$ is in feet / second and t is in seconds.

a. Find the displacement of the object during this time. (i.e., find the net change in the position of the object.)

We know that $v(t) = s'(t)$. We need to find the net change in position, in other words, we need to find $s(3) - s(0)$. By the Net Change Theorem, $s(3) - s(0) = \int_0^3 s'(t)dt$. But . $\int_0^3 s'(t)dt = \int_0^3 v(t)dt = \int_0^3 (30 - 15t)dt$. Now use the Evaluation Theorem (it's OK to use since $30 - 15t$ is continuous on $[0,3]$). $F(t) = 30t - \frac{15}{2}t^2$ is an antiderivative for $30 - 15t$, so

$$\int_0^3 (30 - 15t)dt = F(3) - F(0) = (30 \cdot 3 - \tfrac{15}{2}3^2) - (0 - 0) = 90 - \tfrac{135}{2} = 22.5 \text{ feet}$$

b. Find the total distance travelled by the object on the time interval $[0,3]$.

Here we need to find $\int_0^3 |v(t)|dt$. But $v(t) = 0$ when $t = 2$; $v(t) \geq 0$ on the interval $[0,2]$, and $v(t) \leq 0$ on the interval $[2,3]$. So

$$\int_0^3 |v(t)|dt. = \int_0^2 v(t)dt + \int_2^3 -v(t)dt. = \int_0^2 (30 - 15t)dt + \int_2^3 -(30 - 15t)dt. = \int_0^2 (30 - 15t)dt + \int_2^3 (-30 + 15t)dt.$$

We can evaluate these individually.

$$\int_0^2 (30 - 15t)dt = (30t - \tfrac{15}{2}t^2)\ \Big|_0^2 = (60 - 30) - 0 = 30 \text{ feet, while}$$

$$\int_2^3 (-30 + 15t)dt = (-30t + \tfrac{15}{2}t^2)\ \Big|_2^3 = (-90 + \tfrac{135}{2}) - (-60 + 30) = (-22.5) - (-30) = 8.5 \text{ feet}$$

So the total distance travelled is $30 + 8.5 = 38.5$ feet.

c. State in words the distinction between the interpretations of your answers to parts (a) and (b)

In (a) we find the difference between starting and ending points. The object winds up 22.5 feet away from where it started. In (b) we have figured out exactly how far the object has gone in total. In fact it goes 30 feet in on direction, then 8.5 feet in the opposite direction. So it has gone 38.5 feet in total. (That's what an odometer attached to this object would read.)

8. (2 pt) PRECISELY state the Fundamental Theorem of Calculus, Part 1. If f is continuous on $[a,b]$, then

the function $g(x) = \int_0^x f(t)dt$ has $g'(x) = f(x)$; that is, $g(x)$ is an antiderivative for f.

9. (5 pt total) Let $g(x) = \int_0^x (t - t^2)dt$

a. Find $g'(x)$ by using Part 1 of the Fundamental Theorem of Calculus. (Explain what you are doing.)

Part 1 of the FTC says that $g'(x) = x - x^2$. (The derivative of the definite integral from a to x is the original function of x.)

b. Find $g'(x)$ by evaluating the integral (using the Evaluation Theorem), and then computing the derivative directly using the usual rules for derivatives. (Explain what you are doing.)

$F(t) = \frac{1}{2}t^2 - \frac{1}{3}t^3$ is an antiderivative for $f(t) = t - t^2$. So by the Evaluation Theorem,

$$g(x) = (\tfrac{1}{2}t^2 - \tfrac{1}{3}t^3)\ \Big|_0^x = (\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3) - (0 - 0) = \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3.$$

But then by the usual derivative rules we have $g'(x) = x - x^2$.

10. (3 pt) Let $h(x) = \int_0^{x^3} \sin(t^2+1)dt$. Use Part 1 of the Fundamental Theorem of Calculus to find $h'(x)$.

By the chain rule, $h'(x) = \sin((x^3)^2+1) \cdot \frac{d}{dx}(x^3) = \sin(x^6+1) \cdot 3x^2$.

11. (5 pt each) Evaluate each of the given indefinite integrals. (You may wish to check your answers ...)

a. $\int xe^{-x^2}dx$

u substitution. Let $u = -x^2$. Then $du = -2xdx$, so $\frac{-1}{2}du = xdx$. Now substitute:

$$\int xe^{-x^2}dx = \int e^u \cdot \frac{-1}{2}du = \frac{-1}{2}\int e^u du = \frac{-1}{2}e^u + C = \frac{-1}{2}e^{-x^2} + C.$$

b. $\int \frac{dx}{3-5x}$

u substitution. Let $u = 3-5x$. Then $du = -5dx$, so $\frac{-1}{5}du = dx$. Now substitute:

$$\int \frac{dx}{3-5x} = \int \frac{\frac{-1}{5}du}{u} = \frac{-1}{5}\int \frac{du}{u} = \frac{-1}{5}\ln(u) + C = \frac{-1}{5}\ln(3-5x) + C.$$

c. $\int \sin(2x)dx$

u substitution. Let $u = 2x$. Then $du = 2dx$, so $\frac{1}{2}du = dx$. Now substitute:

$$\int \sin(2x)dx = \int \sin(u)\frac{1}{2}du = \frac{1}{2}\int \sin(u)du = \frac{1}{2}(-\cos(u)) + C = \frac{-1}{2}\cos(2x) + C.$$

12. (1 pt each) True / False

FALSE a. For any function f which is continuous on $[a,b]$ the expression $\int_a^b f(x)dx$ equals the area between the graph of f, the x-axis, and the values $x = a$ and $x = b$. (This is only true when the function $f(x)$ is positive between a and b.

FALSE b. If $v(t)$ represents the velocity of an object on the time interval $[a,b]$, then $\int_a^b v(t)dt$ represents the total distance traveled by the object on the time interval $[a,b]$. $\int_a^b v(t)dt$ represents displacement. If the object always travels in the direction of increasing $s(t)$ on the interval, then this is also distance travelled, but not in general.

TRUE c. If f is continuous on $[a,b]$, then f has an antiderivative on $[a,b]$. The Fundamental Theorem of Calculus Part 1 guarantees it.

FALSE d. Suppose f is continuous on $[a,b]$. If we can't find a nice formula for an antiderivative for f, then the definite integral $\int_a^b f(x)dx$ is not defined. The definite integral is ALWAYS defined. It's just that we need tools other than the Evaluation Theorem to find its value in case we don't know a nice antiderivative for f.

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13. (2 pt each) Short Answer:

a. Give an antiderivative for $f(x) = e^{-x^2}$.

By the Fundamental Theorem of Calculus Part 1, $F(x) = \int_0^x e^{-t^2} dt$is an antiderivative for $f(x)$.

b. It's easy to see that $F(x) = \frac{-1}{x}$ is an antiderivative for $\frac{1}{x^2}$.

b1. Give an example of two numbers a, b for which $\int_a^b \frac{1}{x^2} dx = \frac{-1}{b} - (\frac{-1}{a})$. $a = 1$ $b = 2$.

(The function $\frac{1}{x^2}$ is continuous on $[1, 2]$, so the Evaluation Theorem can be used.)

b2. Give an example of two numbers a, b for which $\int_a^b \frac{1}{x^2} dx \neq \frac{-1}{b} - (\frac{-1}{a})$. $a = -1$ $b = 1$

(The function $\frac{1}{x^2}$ is not continuous on $[-1, 1]$ (because it's not defined at $x = 0$), so the Evaluation Theorem can be used.)

c. How would you go about finding the numerical value of the definite integral $\int_0^1 e^{-x^2} dx$

Since we don't know a nice form for an antiderivative of e^{-x^2}, we just ask our computer to approximate the value by breaking $[0, 1]$ into a lot of pieces (maybe 10,000 or so), and then compute a Riemann sum for that.