Math 414 Fall 2009 Abrams **Practice Exam for Exam 2**

Show all your work. No credit will be given for answers which are not accompanied by supporting computations / justifications. Use the back of the sheet if you need more space. Circle answers when appropriate. Good luck !

1. (2 pt each) SHORT ANSWER.

(a) Let $G$ be a finite group, and let $H$ be a subgroup of $G$. Then Lagrange's Theorem says:

(b) List all the elements of $\mathbf{Z}_2 \times \mathbf{Z}_8$ which have order 2.

(c) List the three abelian groups (up to isomorphism) having order 8.

2. (3 pt total) ONE of parts (a) and (b) of this question has an answer. The other one can't be done. Answer the appropriate one, and write "Can't be done" for the other one.

(a) Give an example of a non-abelian group $G$ and an abelian group $G'$ and an ONTO homomorphism $\varphi : G \to G'$.

(b) Give an example of a non-abelian group $G$ and an abelian group $G'$ and a ONE-to-ONE homomorphism $\varphi : G \to G'$.

3. (1 pt each) Suppose $H$ is a subgroup of $G$, $|H| = 6$, and $a, b \in G$.

(a) How many group operations must be performed to compute the set product $Ha * Hb$ of the two right cosets $Ha$ and $Hb$? (i.e., if you were to compute the set product $Ha * Hb$, how many 'products' would you need to compute?)

(b) If $H$ is a normal subgroup of $G$, how many group elements are in the set $Ha * Hb$?

4. (3 pt total) Find all abelian groups (up to isomorphism) having order 400. Express these groups as direct products of groups of the form $\mathbf{Z}_{p^r}$ where $p$ is prime and $r$ is an integer $\geq 1$. Do not list more than one isomorphic copy of any group.

5. (8 pt total) (a) Let $H$ be a subgroup of the group $G$. There are many statements which are equivalent to the statement "$H$ is a normal subgroup of $G$" (you listed many of them in Quiz 3.) Give the precise statement which is typically the one that we use most often to actually show that a subgroup $H$ is normal in $G$.

(b) Using your answer to (a), prove: Let $G$ denote the group $GL(n, \mathbf{C})$ of invertible $n \times n$ matrices having entries in the complex numbers. Let $H$ denote the subgroup $\{M \in G \mid \det(M) = 1, -1, i,$ or $-i\}$. Prove that $H$ is a normal subgroup of $G$. (You may assume that $H$ is already known to be a subgroup; you need only prove 'normal'.)

(c) Using your answer to (a), prove: Let $G$ and $G'$ be groups, and let $\varphi : G \to G'$ be a homomorphism. Prove that $Ker(\varphi)$ is a normal subgroup of $G$. (You may assume that $Ker(\varphi)$ is already known to be a subgroup; you need only prove 'normal'.)

6. (5 pt) Let $H$ be a subgroup of the group $G$. Let $a, b \in G$. Suppose there is some element $z \in G$ with the property that $z \in aH \cap bH$. Prove that $aH \subseteq bH$. (In fact, $aH = bH$, but you need not show that here. Also, do this problem DIRECTLY; that is, do NOT simply quote a theorem.)

7. (8 pt total) Let $G = \mathbf{Z}_4 \times \mathbf{Z}_8$. Let $H = < (1,6) >$ in $G$.

(a) $|G| =$

(b) List out the elements of $H$.

(c) Since $G$ is abelian, we can form the factor group $G/H$. $|G/H| =$

(d) List the elements of the factor group $G/H$. Do not list any element more than once.

(e) For each of the elements you listed in part (d), give the order of the element in $G/H$..

(f) To what 'known' abelian group is $G/H$ isomorphic? Justify your answer.

8. (5 pt total) Let $G$ denote the group $GL(n, \mathbf{R})$ of invertible $n \times n$ matrices having entries in the real numbers. (So the operation is matrix multiplication.). Define the function $\varphi : G \to \mathbf{Z}_2$ by setting $\varphi(M) = 0$ in case $\det(M) > 0$, and $\varphi(M) = 1$ in case $\det(M) < 0$

(a) Prove that $\varphi$ is a group homomorphism.

(b) To what group is $G/Ker\varphi$ isomorphic?

9 (1 pt each) TRUE / FALSE.

(a) **T** **F** $\mathbf{Z}_2 \times \mathbf{Z}_8$ is an abelian group.

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(b) **T** **F** If $H$ is a normal subgroup of the group $G$, then $a^{-1}ha = h$ for each $a \in G$ and $h \in H$.

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(c) **T** **F** If $G$ is a group, $H \leq G$, and $a \in G$, then $Ha = H$ if and only if $a \in H$.

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(d) **T** **F** If $H$ is a normal subgroup of $G$, then $H$ is abelian

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(e) **T** **F** If $G$ is abelian, then every subgroup of $G$ is normal..

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(f) **T** **F** If $H$ is a subgroup of $G$, and $H$ is abelian, then $H$ is a normal subgroup of $G$.

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(g) **T** **F** If $G$ is a group having seven elements, then $G$ is cyclic.

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(h) **T** **F** There exist groups $G$ and $G'$ for which $G$ is a subgroup of $(\mathbf{Z},+)$, $G'$ is a subgroup of $(\mathbf{C}^*,\cdot)$, and $G$ is isomorphic to $G'$.

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(i) **T** **F** If $H$ is a subgroup of the finite group $G$, and $a \in G$, then the left coset $aH$ contains the same elements as the right coset $Ha$.

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(j) **T** **F** Suppose $\varphi : G \to G'$ is a group homomorphism. Then $Ker(\varphi)$ is a normal subgroup of $G$.

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(k) **T** **F** Suppose $H$ is a normal subgroup of $G$. Then there is some group $G'$, and a homomorphism $\varphi : G \to G'$, for which $H = Ker(\varphi)$.

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(l) **T** **F** There exists an onto homomorphism from $\mathbf{Z}$ to $\mathbf{Z}_4$.