Comparing and contrasting algebraic graph $K$-algebras with graph $C^*$-algebras

Gene Abrams

University of Colorado at Colorado Springs

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Overview

1. Leavitt path algebras
2. Connections to graph $C^*$-algebras
3. Similarities
4. Differences
5. Similar or Different?

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1 Leavitt path algebras

2 Connections to graph C*-algebras

3 Similarities

4 Differences

5 Similar or Different?
General path algebras

\( K \) always denotes a field.

Let \( E \) be a directed graph. \( E = (E^0, E^1, r, s) \)

\[ \bullet s(e) \xrightarrow{e} \bullet r(e) \]

The *path algebra* \( KE \) is the \( K \)-algebra with basis \( \{ p_i \} \) consisting of the directed paths in \( E \). (View vertices as paths of length 0.)

\[ p \cdot q = pq \quad \text{if} \quad r(p) = s(q), \quad 0 \quad \text{otherwise}. \]

In particular, \( s(e) \cdot e = e = e \cdot r(e) \).

Note: \( E^0 \) finite \( \iff \) \( KE \) is unital; then \( 1_{KE} = \sum_{v \in E^0} v \).
Building Leavitt path algebras

Start with $E$, build its *double graph* $\hat{E}$. 
Building Leavitt path algebras

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\[ E = \]

\[ \begin{array}{cccc}
v & w & t & u \\
e & h \\
\downarrow & \downarrow \\
v & w & t & u \\
\end{array} \]

\[ \begin{array}{cccc}
w & x & t & u \\
\downarrow & \downarrow & \downarrow & \downarrow \\
w & x & t & u \\
\end{array} \]

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Building Leavitt path algebras

Start with $E$, build its double graph $\hat{E}$. Example:

$$E =$$

$$\hat{E} =$$
Building Leavitt path algebras

Construct the path algebra $K\hat{E}$. 

**Definition**

The Leavitt path algebra of $E$ with coefficients in $K$ is

$$L_K(E) = K\hat{E}/\langle (CK_1), (CK_2) \rangle$$
Building Leavitt path algebras

Construct the path algebra $K\hat{E}$. Consider these relations in $K\hat{E}$:
Building Leavitt path algebras

Construct the path algebra $K\hat{E}$. Consider these relations in $K\hat{E}$:

(CK1) $e^*e = r(e)$ for all $e \in E^1$; $f^*e = 0$ for all $f \neq e \in E^1$.

(CK2) $v = \sum\{e \in E^1 | s(e) = v\} ee^*$ for all $v \in E^0$
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Construct the path algebra $K\hat{E}$. Consider these relations in $K\hat{E}$:

(CK1) $e^*e = r(e)$ for all $e \in E^1$; $f^*e = 0$ for all $f \neq e \in E^1$.

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(just at regular vertices $\nu$, i.e., not sinks, not infinite emitters)
Building Leavitt path algebras

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Definition

The Leavitt path algebra of $E$ with coefficients in $K$

$L_K(E) = K\hat{E} / < (CK1), (CK2) >$
Leavitt path algebras: Examples

Some sample computations in $L_{\mathbb{C}}(E)$ from the Example:

\[ \hat{E} = \]

\[ ee^* + ff^* + gg^* = v \quad g^*g = w \quad g^*f = 0 \]
\[ h^*h = w \quad hh^* = u \quad ff^* = \ldots \quad \text{(no simplification)} \]
Leavitt path algebras: Examples

Standard algebras arising as Leavitt path algebras:
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\[ E = \bullet v_1 \xrightarrow{e_1} \bullet v_2 \xrightarrow{e_2} \bullet v_3 \ldots \ldots \bullet v_{n-1} \xrightarrow{e_{n-1}} \bullet v_n \]

Then \( L_K(E) \cong M_n(K) \).
Leavitt path algebras: Examples

Standard algebras arising as Leavitt path algebras:

\[ E = \bullet v_1 \xrightarrow{e_1} \bullet v_2 \xrightarrow{e_2} \bullet v_3 \rightarrow \cdots \bullet v_{n-1} \xrightarrow{e_{n-1}} \bullet v_n \]

Then \( L_K(E) \cong M_n(K) \).

\[ E = \bullet v \xrightarrow{\circlearrowright} x \]

Then \( L_K(E) \cong K[x, x^{-1}] \).
Leavitt path algebras: Examples

\[ E = R_n = \]

Then \( L_K(E) \cong L_K(1, n) \), the “Leavitt \( K \)-algebra of order \( n \)”.


\( L_K(1, n) \) is the universal \( K \)-algebra \( R \) for which \( RR \cong RR^n \).
Leavitt path algebras: Examples

\[ E = R_n = \begin{array}{ccc}
  \bullet & \rightarrow & y_1 \\
  & \cdots & \\
  \downarrow & \cdots & \downarrow \\
  \rightarrow & \cdots & \rightarrow \\
  y_3 & \rightarrow & y_2 \\
 \end{array} \]

Then \( L_K(E) \cong L_K(1, n) \), the “Leavitt \( K \)-algebra of order \( n \)”.

\[(W.G. \ Leavitt, \ Transactions. \ A.M.S. \ 1962).\]

\( L_K(1, n) \) is the universal \( K \)-algebra \( R \) for which \( RR \cong RR^n \).

\[ L_K(1, n) = \langle x_1, \ldots, x_n, y_1, \ldots, y_n \mid x_i y_j = \delta_{i,j} 1_K, \sum_{i=1}^{n} y_i x_i = 1_K \rangle \]
Leavitt path algebras

Some general properties of Leavitt path algebras:

1. $L_K(E) = \text{span}_K \{pq^* \mid p, q \text{ paths in } E \}$.

2. $L_K(E) \cong L_K(E)^{op}$.

3. $L_K(E)$ admits a natural $\mathbb{Z}$-grading: $\deg(pq^*) = \ell(p) - \ell(q)$.

4. $J(L_K(E)) = \{0\}$. 
1. Leavitt path algebras

2. Connections to graph $C^*$-algebras

3. Similarities

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5. Similar or Different?
Graph $C^*$-algebras

$E$ any directed graph, $\mathcal{H}$ a Hilbert space.

**Definition.** A **Cuntz-Krieger $E$-family** in $B(\mathcal{H})$ is a collection of mutually orthogonal projections $\{P_v \mid v \in E^0\}$, and partial isometries $\{S_e \mid e \in E^1\}$ with mutually orthogonal ranges, for which:

(CK1) $S_e^* S_e = P_{r(e)}$ for all $e \in E^1$,

(CK2) $\sum_{\{e \mid s(e) = v\}} S_e S_e^* = P_v$ whenever $v$ is a regular vertex, and

(CK3) $S_e S_e^* \leq P_{s(e)}$ for all $e \in E^1$.

The **graph $C^*$-algebra** $C^*(E)$ of $E$ is the universal $C^*$-algebra generated by a Cuntz-Krieger $E$-family.
Graph $C^*$-algebras

For $\mu = e_1 e_2 \cdots e_n$ a path in $E$,
let $S_\mu$ denote $S_{e_1} S_{e_2} \cdots S_{e_n} \in C^*(E)$.

**Proposition:** Consider

$$A = \text{span}_\mathbb{C} \{ P_\nu, S_\mu S_{\nu}^* \mid \nu \in E^0, \, \mu, \nu \text{ paths in } E^1 \} \subseteq C^*(E).$$

Then $L_\mathbb{C}(E) \cong A$ as $*$-algebras.
Graph C*-algebras

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Consequently, $C^*(E)$ may be viewed as the completion (in operator norm) of $L_\mathbb{C}(E)$.
Graph $C^*$-algebras

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Then $L_\mathbb{C}(E) \cong A$ as $*$-algebras.

Consequently, $C^*(E)$ may be viewed as the completion (in operator norm) of $L_\mathbb{C}(E)$.

So it’s probably not surprising that there are some close relationships between $L_\mathbb{C}(E)$ and $C^*(E)$. 
Here are the graph C*-algebras which arise from the graphs of the previous examples.

\[ E = \bullet v_1 \xrightarrow{e_1} \bullet v_2 \xrightarrow{e_2} \bullet v_3 \quad \ldots \quad \bullet v_{n-1} \xrightarrow{e_{n-1}} \bullet v_n \]

Then \( C^*(E) \cong M_n(\mathbb{C}) \cong L_\mathbb{C}(E) \).
Graph $C^*$-algebras: Examples

Here are the graph $C^*$-algebras which arise from the graphs of the previous examples.

\[ E = \bullet v_1 \xrightarrow{e_1} \bullet v_2 \xrightarrow{e_2} \bullet v_3 \ldots \xrightarrow{e_{n-1}} \bullet v_n \]

Then $C^*(E) \cong M_n(\mathbb{C}) \cong L_\mathbb{C}(E)$.

\[ E = \bullet v \xrightarrow{\circlearrowleft} \]

Then $C^*(E) \cong C(\mathbb{T})$, the continuous functions on the unit circle.
Graph $C^*$-algebras: Examples

Then $C^*(E) \cong \mathcal{O}_n$, the Cuntz algebra of order $n$. 
Some graph terminology

Example

1 cycle;
Some graph terminology

Example

1 cycle; exit for a cycle;
Some graph terminology

Example

1 cycle; exit for a cycle; Condition (L)
Some graph terminology

Example

1. **cycle**; **exit for a cycle**; **Condition (L)**

2. **Condition (K)**
Some graph terminology

Example

1. cycle; exit for a cycle; Condition (L)

2. Condition (K); Note that $v \bigcirc$ does not have (K)
Some graph terminology

Example

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3. downward directed
Some graph terminology

Example

- cycle; exit for a cycle; Condition (L)
- Condition (K); Note that \( v \) does not have (K)
- downward directed (also called Condition (MT3))
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Example

1. cycle; exit for a cycle; Condition (L)
2. Condition (K); Note that \( v \) does not have (K)
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Example

1. cycle; exit for a cycle; Condition (L)
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Example

1. cycle; exit for a cycle; Condition (L)
2. Condition (K); Note that \( v \) does not have (K)
3. downward directed (also called Condition (MT3))
4. connects to a cycle; cofinal

Standing hypothesis: All graphs are finite (for now) ...
1. Leavitt path algebras

2. Connections to graph $C^*$-algebras

3. Similarities

4. Differences

5. Similar or Different?
We begin by looking at some similarities between the structure of $L_K(E)$ and the structure of $C^*(E)$. 
Simplicity

Simplicity:

*Algebraic:* No nontrivial two-sided ideals.

*Analytic:* No nontrivial closed two-sided ideals.
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*Algebraic:* No nontrivial two-sided ideals.

*Analytic:* No nontrivial closed two-sided ideals.

*Result:* These are equivalent for any finite graph $E$:

1. $L_C(E)$ is simple
2. $L_K(E)$ is simple for any field $K$
3. $C^*(E)$ is (topologically) simple
4. $C^*(E)$ is (algebraically) simple
5. $E$ is cofinal, and satisfies Condition (L).
The $\mathcal{V}$-monoid:

**Algebraic:** For a ring $R$, $\mathcal{V}(R)$ is the monoid of isomorphism classes of finitely generated left $R$-modules, with operation $\oplus$. $\mathcal{V}(R)$ can be viewed as the set of equivalence classes $V(e)$ of idempotents $e$ in the (nonunital) infinite matrix ring $M_\infty(R)$, with operation

$$V(e) + V(f) = V(\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}).$$

**Analytic:** For an operator algebra $A$, $\mathcal{V}_{MVN}(A)$ is the monoid of Murray - von Neumann equivalence classes of projections in $M_\infty(A)$.
The $\mathcal{V}$-monoid

Whenever $A$ is a $C^*$-algebra, then $\mathcal{V}(A)$ agrees with $\mathcal{V}_{MvN}(A)$.

The natural inclusion $\psi : L_C(E) \to C^*(E)$ induces a monoid isomorphism $\mathcal{V}(\psi) : \mathcal{V}(L_C(E)) \to \mathcal{V}(C^*(E))$. 
The $\mathcal{V}$-monoid

Whenever $A$ is a C*-algebra, then $\mathcal{V}(A)$ agrees with $\mathcal{V}_{MvN}(A)$.

The natural inclusion $\psi : L_{\mathbb{C}}(E) \to C^*(E)$ induces a monoid isomorphism $\mathcal{V}(\psi) : \mathcal{V}(L_{\mathbb{C}}(E)) \to \mathcal{V}(C^*(E))$. Moreover, The monoid $\mathcal{V}(L_K(E))$ is independent of the field $K$; specifically, $\mathcal{V}(L_K(E)) \cong M_E$, the graph monoid of $E$. 
The $\mathcal{V}$-monoid

Whenever $A$ is a $C^*$-algebra, then $\mathcal{V}(A)$ agrees with $\mathcal{V}_{MvN}(A)$.

The natural inclusion $\psi : L_{\mathbb{C}}(E) \to C^*(E)$ induces a monoid isomorphism $\mathcal{V}(\psi) : \mathcal{V}(L_{\mathbb{C}}(E)) \to \mathcal{V}(C^*(E))$. Moreover, The monoid $\mathcal{V}(L_K(E))$ is independent of the field $K$; specifically, $\mathcal{V}(L_K(E)) \cong M_E$, the graph monoid of $E$.

**Result:** For any finite graph $E$ and any field $K$, the following semigroups are isomorphic.

1. The graph monoid $M_E$
2. $\mathcal{V}(L_K(E))$
3. $\mathcal{V}(C^*(E))$
4. $\mathcal{V}_{MvN}(C^*(E))$. 

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Purely infinite simplicity

**Purely infinite simplicity:**

*Algebraic:* $R$ is purely infinite simple in case $R$ is simple and every nonzero right ideal of $R$ contains an infinite idempotent.

*Analytic:* The simple $C^*$-algebra $A$ is called purely infinite (simple) if for every positive $x \in A$, the subalgebra $xAx$ contains an infinite projection.
Purely infinite simplicity:

**Algebraic**: $R$ is purely infinite simple in case $R$ is simple and every nonzero right ideal of $R$ contains an infinite idempotent.

**Analytic**: The simple $C^*$-algebra $A$ is called purely infinite (simple) if for every positive $x \in A$, the subalgebra $xAx$ contains an infinite projection.

(Algebraic) purely infinite simplicity is equivalent to: $R$ is not a division ring, and for all nonzero $x \in R$ and all $y \in R$ there exist $\alpha, \beta \in R$ for which $y = \alpha x \beta$.

(Topological) purely infinite simplicity is equivalent to: $A \neq \mathbb{C}$ and for every $x \neq 0$ in $A$ there exist $\alpha, \beta \in A$ for which $\alpha x \beta = 1$. 

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Purely infinite simplicity

**Result:** These are equivalent for a finite graph $E$:

1. $L_{\mathbb{C}}(E)$ is purely infinite simple.
2. $L_K(E)$ is purely infinite simple for any field $K$.
3. $C^*(E)$ is (topologically) purely infinite simple.
4. $C^*(E)$ is (algebraically) purely infinite simple.
5. $E$ is cofinal, every cycle in $E$ has an exit, and every vertex in $E$ connects to a cycle.
Cuntz-Krieger Uniqueness Theorem

**Theorem**: (Analytic) Suppose $E$ is a graph satisfying Condition (L). Suppose $\{T, Q\}$ is a Cuntz-Krieger $E$-family in a C*-algebra $B$ for which $Q_v \neq 0$ for all $v \in E^0$.

Then the canonical homomorphism $\pi_{T,Q} : C^*(E) \to B$ is injective.
**Cuntz-Krieger Uniqueness Theorem**

**Theorem:** (Analytic) Suppose $E$ is a graph satisfying Condition (L). Suppose $\{T, Q\}$ is a Cuntz-Krieger $E$-family in a $C^*$-algebra $B$ for which $Q_v \neq 0$ for all $v \in E^0$.

Then the canonical homomorphism $\pi_{T,Q} : C^*(E) \rightarrow B$ is injective.

**Theorem:** (Algebraic) Suppose $E$ is a graph satisfying Condition (L). Let $K$ be a field, and let $R$ be any $K$-algebra. Suppose $\theta : L_K(E) \rightarrow R$ is a homomorphism of algebras for which $\theta(v) \neq 0$ for all $v \in E^0$.

Then $\theta$ is injective.
Primitivity

**Primitivity:**

*Algebraic:* $R$ is (left) primitive if there exists a simple faithful left $R$-module.

*Analytic:* $A$ is (topologically) primitive if there exists a faithful irreducible representation $\pi : A \to B(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. 

Result: These are equivalent for a finite graph $E$:

1. $L_C(E)$ is primitive.
2. $L_K(E)$ is primitive for any field $K$.
3. $C^*(E)$ is (topologically) primitive.
4. $C^*(E)$ is (algebraically) primitive.
5. $E$ is downward directed and satisfies Conditions (L).
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1. Leavitt path algebras

2. Connections to graph $C^*$-algebras

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We now look at some differences between the structure of $L_K(E)$ and the structure of $C^*(E)$. 
Primeness:

*Algebraic:* $R$ is a prime ring in case $\{0\}$ is a prime ideal of $R$; that is, in case for any two-sided ideals $I, J$ of $R$, $I \cdot J = \{0\}$ if and only if $I = \{0\}$ or $J = \{0\}$.
Primeness

**Theorem.** $K$ any field, $E$ any graph.

$L_K(E)$ is prime $\iff E$ is downward directed.
Primeness

**Theorem.** $K$ any field, $E$ any graph.

$L_K(E)$ is prime $\iff$ $E$ is downward directed.

**Sketch of proof.** Suppose $R = L_K(E)$ is a prime; let $v, w \in E^0$.

(1) Since $RvR$ and $RwR$ are nonzero, so is $RvRwR$, and hence $vRw$.

(2) So $v\alpha\beta^*w \neq 0$ for some paths $\alpha, \beta$ in $E$, which in particular gives $s(\alpha) = v$ and $s(\beta) = w$.

(3) Then $u = r(\alpha) = r(\beta)$ is a vertex which works.
Conversely, suppose $E$ is downward directed.

(1) Since $L_K(E)$ is $\mathbb{Z}$-graded, it suffices to show $IJ \neq \{0\}$ for any pair $I, J$ of nonzero graded ideals of $L_K(E)$.

(2) Every nonzero graded ideal of $L_K(E)$ contains a vertex.

(3) Let $v \in I \cap E^0$ and $w \in J \cap E^0$. There is a vertex $u$, and paths $p, q$ in $E$ such that $p : v \rightsquigarrow u$ and $q : w \rightsquigarrow u$.

(4) But then $u = p^*vp \in I$ and $u = q^*wq \in J$, so that $0 \neq u = u^2 \in IJ$. 
Primeness

Analytic: A is a prime C*-algebra in case \( \{0\} \) is a prime ideal of \( A \); that is, in case for any closed two-sided ideals \( I, J \) of \( R \), \( I \cdot J = \{0\} \) if and only if \( I = \{0\} \) or \( J = \{0\} \).

Remark: Since \( I \cdot J = \{0\} \) implies \( \overline{I} \cdot \overline{J} = \{0\} \), we get that \( A \) is algebraically prime if and only if \( A \) is analytically prime.

But any separable C*-algebra is (topologically) prime if and only if it is (topologically) primitive. So in particular for finite \( E \), \( C^*(E) \) is prime \( \iff \) if \( C^*(E) \) is primitive, so that (above)

\[ C^*(E) \text{ prime } \iff E \text{ downward directed and satisfies Condition (L)}. \]
So for example $L_K(\bullet \xRightarrow{\_\_\_} )$ is prime, but $C^*(\bullet \xRightarrow{\_\_\_} )$ is not prime.
So for example $L_K(\bullet \xrightarrow{\circ} \bullet)$ is prime, but $C^*(\bullet \xrightarrow{\circ} \bullet)$ is not prime.

Question: Is there a 'natural' analytic property $\mathcal{P}$ on $C^*$-algebras which would yield a result of the form:

$$C^*(E) \text{ has property } \mathcal{P} \text{ if and only if } E \text{ is downward directed?}$$
So for example $L_K(\bullet \xrightarrow{\bullet} \bigcirc)$ is prime, but $C^*(\bullet \xrightarrow{\bullet} \bigcirc)$ is not prime.

Question: Is there a 'natural' analytic property $P$ on $C^*$-algebras which would yield a result of the form:

$$C^*(E) \text{ has property } P \text{ if and only if } E \text{ is downward directed?}$$

This could give some sort of natural connection between graph $C^*$-algebras and the algebraic notion of primeness.
Primeness

We can describe all the graded prime ideals of $L_K(E)$ in terms of subsets of $E$, and can distinguish primitive ones from nonprimitive.

We also have:

Theorem (Chris Smith, in preparation) Given a finite partially ordered set $X$, there exists a finite graph $E$ for which $\text{grSpec}(L_K(E)) \sim = X$. 

... and the primitives can be embedded as desired.

As well as:

Theorem For any finite graph $E$, $L_K(E)$ satisfies the Dixmier-Moeglin equivalence.
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As well as:

**Theorem**

For any finite graph $E$, $L_K(E)$ satisfies the Dixmier-Moeglin equivalence.
Tensor products of graph algebras

Well known: $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. 
Well known: $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.

Question: Is the analogous statement true for Leavitt path algebras? i.e., do we have

$$L_K(1, 2) \otimes_K L_K(1, 2) \cong L_K(1, 2) ?$$

Open for about five years.
Tensor products of graph algebras

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Question: Is the analogous statement true for Leavitt path algebras? i.e., do we have

\[
L_K(1, 2) \otimes_K L_K(1, 2) \cong L_K(1, 2) ?
\]

Open for about five years.

Then (early 2011) Answer: No.

Ara & Cortiñas; Dicks; Bell & Bergman
Using Ara / Cortiñas approach, it follows that

\[ \otimes^s L_K(1, 2) \cong \otimes^t L_K(1, 2) \iff s = t. \]
Tensor products of graph algebras

Using Ara / Cortiñas approach, it follows that

$$\otimes^s L_K(1, 2) \cong \otimes^t L_K(1, 2) \iff s = t.$$  

Using Dicks’ approach, we can show

**Proposition.** For finite graphs $E, F$,

$$L_K(E) \otimes L_K(F) \cong L_K(G) \text{ some } G \iff \text{ at least one of } E, F \text{ is acyclic}$$
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\[ L_K(E) \otimes L_K(F) \cong L_K(G) \iff E \text{ or } F \text{ acyclic} \]

**Sketch of Proof.**

1. For any finite \( E \), \( L_K(E) \) has \( \text{proj.dim.}(L_K(E)) \leq 1 \).
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\[ L_K(E) \otimes L_K(F) \cong L_K(G) \iff E \text{ or } F \text{ acyclic} \]

**Sketch of Proof.**

1. For any finite $E$, $L_K(E)$ has $\text{proj.dim.}(L_K(E)) \leq 1$.

2. $L_K(E)$ is von Neumann regular $\iff E$ is acyclic.
   
   ($\text{vNr} \iff \text{every } R\text{-module is flat} \iff \forall a \in R \exists x \in R, a = axa.$)
**Sketch of Proof.**

1. For any finite $E$, $L_K(E)$ has $\text{proj.dim.}(L_K(E)) \leq 1$.
2. $L_K(E)$ is von Neumann regular $\iff E$ is acyclic.
   
   (vNr $\iff$ every $R$-module is flat $\iff \forall a \in R \exists x \in R, a = axa$.)
3. So $\text{flatdim.}(L_K(E)) = 1 \iff E$ contains a cycle.
$L_K(E) \otimes L_K(F) \cong L_K(G) \iff E \text{ or } F \text{ acyclic}$

**Sketch of Proof.**

1. For any finite $E$, $L_K(E)$ has $\text{proj.dim.}(L_K(E)) \leq 1$.
2. $L_K(E)$ is von Neumann regular $\iff E$ is acyclic.
   \( (\text{vNr} \iff \text{every } R\text{-module is flat} \iff \forall a \in R \ \exists x \in R, a = axa. ) \)
3. So $\text{flatdim.}(L_K(E)) = 1 \iff E$ contains a cycle.
4. Old result of Eilenberg et. al.: For $K$-algebras $A, B$,
   \[ \text{proj.dim.}(A) + \text{flatdim.}(B) \leq \text{proj.dim.}(A \otimes B). \]
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5. So if both $E$ and $F$ contain a cycle, then
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Leavitt path algebras
Connections to graph C*-algebras
Similarities
Differences
Similar or Different?

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   $\text{proj.dim.}(L_K(E) \otimes L_K(F)) \geq 2$.
6. If one of $E, F$ is acyclic (say $E$), then $L_K(E) \otimes L_K(F)$ is a
direct sum of full matrix rings over $L_K(F)$. 
1. Leavitt path algebras

2. Connections to graph C*-algebras

3. Similarities

4. Differences

5. Similar or Different?
Similarities

We continue by looking at properties for which we do not currently know whether these give similarities or differences between the structure of $L_K(E)$ and the structure of $C^*(E)$. 

Gene Abrams

Comparing and contrasting algebraic graph $K$-algebras with graph $C^*$-algebras

Leavitt path algebras
Connections to graph $C^*$-algebras
Similarities
Differences
Similar or Different?
The isomorphism question

Perhaps the most basic question ...

If $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$, does this imply $C^*(E) \cong C^*(F)$?

And conversely?

(Need to interpret “isomorphism” appropriately.)
The isomorphism question

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And conversely?

(Need to interpret “isomorphism” appropriately.)

Partial answer: OK in case the graph algebras are simple.
(This uses classification results.)

Answer not known in general.
An algebraic Kirchberg / Phillips Theorem?

Suppose $E$ and $F$ are finite graphs for which $C^*(E)$ and $C^*(F)$ (equivalently, $L_C(E)$ and $L_C(F)$) are simple. Assume that these are also purely infinite.

Well-known (and deep):

If $(K_0(C^*(E)), [1_{C^*(E)}]) \cong (K_0(C^*(F)), [1_{C^*(F)}])$, then $C^*(E) \cong C^*(F)$. 

Gene Abrams
Comparing and contrasting algebraic graph $K$-algebras with graph $C^*$-algebras.
An algebraic Kirchberg / Phillips Theorem?

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An algebraic Kirchberg / Phillips Theorem?

One approach:

1. Use results from symbolic dynamics to show that the isomorphism $C^*(E) \cong C^*(F)$ follows in case one also assumes that $\det(I - A_E) = \det(I - A_F)$.

2. Use $KK$-theory to show that the graph $C^*$-algebras of these two graphs $E$, $E_-$ are isomorphic:

```
• v₁ ← v₂
  v₃ ← v₄
```

(These have identical $K$-theory, but different determinants.)

3. Reduce the “bridging of the determinant gap” for all appropriate pairs of graphs to the question of establishing a specific isomorphism of an infinite dimensional vector space having specified properties (use the isomorphism from (2))

4. Show such an isomorphism exists.
An algebraic Kirchberg / Phillips Theorem?

A second approach:

Use the Kirchberg / Phillips Theorem.
An algebraic Kirchberg / Phillips Theorem?

For Leavitt path algebras we have:

1. Using results from symbolic dynamics, can show that the isomorphism $L_K(E) \cong L_K(F)$ follows in case one also assumes that $\det(I - A_E) = \det(I - A_F)$.

2. We do not know whether $L_K(E) \cong L_K(E_-)$. (Is there a good analog to KK theory in the algebraic context?)

3. Reduce the “bridging of the determinant gap” for all appropriate pairs of graphs to the question of establishing a specific isomorphism of an infinite dimensional vector space having specified properties (assuming one could use the isomorphism from (2), in case such exists)

4. Show such an isomorphism exists.
Kaplansky’s question

Kaplansky: *Is a von Neumann regular prime ring necessarily primitive?*


Answered in the negative (Domanov, 1977), a group-algebra example.
Kaplansky’s question

**Theorem.** (A-, Rangaswamy) Let $E$ be an arbitrary graph. Then $L_K(E)$ is von Neumann regular if and only if $E$ is acyclic.
Kaplansky’s question

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**Theorem.** (A-, Bell, Rangaswamy (in press)) Let $E$ be an arbitrary graph. Then $L_K(E)$ is primitive if and only if

1. $L_K(E)$ is prime (i.e., $E$ is downward directed),
2. $E$ satisfies Condition (L), and
3. there exists a countable set of vertices $S$ in $E$ for which every vertex of $E$ connects to an element of $S$. ("Countable Separation Property")
Kaplansky’s question

It’s not hard to find acyclic graphs $E$ for which $L_K(E)$ is prime but for which C.S.P. fails.

**Example:** $X$ uncountable, $S$ the set of finite subsets of $X$. Define the graph $E$:

1. vertices indexed by $S$, and
2. edges induced by proper subset relationship.

Then $L_K(E)$ is von Neumann regular, and prime, but not primitive.
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2. edges induced by proper subset relationship.

Then $L_K(E)$ is von Neumann regular, and prime, but not primitive.

Note: Adjoining $1_K$ in the usual way (Dorroh extension by $K$) gives unital, regular, prime, not primitive algebras.
Kaplansky’s question

Kaplansky: “It seems unlikely that the answer is affirmative, but a counter-example may have to be weird.”
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Kaplansky: “*It seems unlikely that the answer is affirmative, but a counter-example may have to be weird.*”

Or maybe not so weird?
Kaplansky’s question

Kaplansky: “It seems unlikely that the answer is affirmative, but a counter-example may have to be weird.”

Or maybe not so weird?

Remark: These examples are actually “Cohn algebras”.
Kaplansky’s question

Question (in progress): Does the Countable Separation Property (with together with downward directedness and Condition (L)) also classify the primitive graph C*-algebras?
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These don’t include infinite acyclic graphs.
Kaplansky’s question

Question (in progress): Does the Countable Separation Property (with together with downward directedness and Condition (L)) also classify the primitive graph C*-algebras?

Question (in progress): Which are the von Neumann regular graph C*-algebras?

These don’t include infinite acyclic graphs.

So von Neumann regularity differs between Leavitt path algebras and graph C*-algebras (even in the separable case).
Simple Lie algebras

**Theorem.** (A-, Funk-Neubauer 2011) $K$ any field, $n \geq 2$. Then the Lie algebra $[M_d(L_K(1, n)), M_d(L_K(1, n))]$ is simple
Simple Lie algebras

**Theorem.** (A-, Funk-Neubauer 2011) $K$ any field, $n \geq 2$. Then the Lie algebra $[M_d(L_K(1, n)), M_d(L_K(1, n))]$ is simple $\iff$ char($K$) divides $n - 1$ and char($K$) does not divide $d$. 
Theorem. (A-, Funk-Neubauer 2011)  \( K \) any field, \( n \geq 2 \). Then the Lie algebra \( [M_d(L_K(1, n)), M_d(L_K(1, n))] \) is simple \( \iff \)

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Theorem. (A-, Zak Mesyan (in press))  \( E \) any finite graph. Then there is a straightforward algorithm to determine when \( [L_K(E), L_K(E)] \) is a simple Lie algebra.
Simple Lie algebras

**Theorem.** (A-, Funk-Neubauer 2011) \( K \) any field, \( n \geq 2 \). Then the Lie algebra \([M_d(L_K(1, n)), M_d(L_K(1, n))]\) is simple \( \iff \)

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**Theorem.** (A-, Zak Mesyan (in press)) \( E \) any finite graph. Then there is a straightforward algorithm to determine when \([L_K(E), L_K(E)]\) is a simple Lie algebra.

Proofs of both theorems depend heavily on representations of elements as finite sums of \( pq^* \).

Current investigation: Can these results be extended to Lie algebras of the form \([C^*(E), C^*(E)]\)?
Similar or Different?

Thank you.

Questions?