Math 414 Fall 2009 Abrams **Exam 2** Name ______________________

Show all your work. No credit will be given for answers which are not accompanied by supporting computations / justifications. Use the back of the sheet if you need more space. Circle answers when appropriate. Good luck !

1. (2 pt each) SHORT ANSWER.

(a) Let G be a finite group, and let H be a subgroup of G. Then Lagrange's Theorem says:

(b) Give an example of an infinite group G and a normal subgroup H of G for which the factor group G/H is finite.

$G =$ ________ $H =$ ________

2. (2 pt total) For each of these two statements, indicate which of the three possible responses best describes it.

(a) _____ For each finite group G and subgroup H of G, the number of left cosets of H in G equals the number of right cosets of H in G.

(b) ______ For each finite group G subgroup H of G, and element a of G, the left coset aH equals the right coset Ha.

Possible responses:

(i) This is always true.

(ii) This is not always true, but is true in the particular group $G = \mathbf{Z}_4$.

(iii) This is not always true, but is true in the particular group $G = D_4$.

3. (7 pt total) Let G denote the group $GL(3,\mathbf{Q})$ of invertible 3×3 matrices having entries in the rational numbers. (So the operation is matrix multiplication.). Define the function $\varphi : G \to (\mathbf{Q}^*, \cdot)$ by setting $\varphi(M) = 1$ in case $\det(M) > 0$, and $\varphi(M) = -1$ in case $\det(M) < 0$

(a) Prove that φ is a homomorphism. [Possible hint: four cases.]

(b) Give an example of an element of G other than I_3 which is clearly in $Ker(\varphi)$.

(c) Give an example of an element of G which is clearly not in $Ker(\varphi)$.

(d) **T** **F** φ is one-to-one.

4. (5 pt total) Let G be a cyclic group, and let H be a subgroup of G.

(a) H is necessarily a normal subgroup of G. Why?

(b) Prove that that G/H is a cyclic group.

5. (5 pt total) (a) Let H be a subgroup of the group G. There are many statements which are equivalent to the statement "H is a normal subgroup of G" (you listed many of them in Quiz 3.) Precisely give the statement which is typically the most useful one to use in actually showing a subgroup H is normal in G.

(b) Using your answer to (a), prove: Let G and G' be groups, and let $\varphi : G \to G'$ be a homomorphism. Prove that $Ker(\varphi)$ is a normal subgroup of G. (You may assume that $Ker(\varphi)$ is already known to be a subgroup; you need only prove 'normal'.)

6. (5 pt) Let H be a subgroup of the group G. Let $a, b \in G$. Suppose there is some element $z \in G$ with the property that $z \in aH \cap bH$. Prove that $aH \subseteq bH$. (In fact, $aH = bH$, but you need not show that here. Also, do this problem DIRECTLY; that is, do NOT simply quote a theorem.)

7. (8 pt total) Let $G = \mathbf{Z}_2 \times \mathbf{Z}_6$. Let $H = <(0,2)>$ in G.

(a) $|G| =$

(b) List out the elements of H.

(c) Since G is abelian, we can form the factor group G/H. $|G/H| =$

(d) List the elements of G/H. Do not list any element more than once.

(e) Give the group table of G/H.

(f) **T** **F** G/H is cyclic.

(g) To what 'known' abelian group is G/H isomorphic? Justify your answer.

8. (5 pt total) (a) Let $G = (\mathbf{C}^*, \cdot)$ be the group of nonzero complex numbers under multiplication. Let G' denote the subgroup of $GL(2, \mathbf{R})$ consisting of matrices of the form $\left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbf{R}, \text{ where at least one of } a \text{ or } b \text{ is nonzero} \right\}$. (So the operation in G' is matrix multiplication.) Prove that G is isomorphic to G'.

(b) Give an example of an element $k = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$ in G' which has the property that $k^2 \neq e'$, but $k^4 = e'$.

9. (3 pt) Give a list of four groups, none of which is isomorphic to another, where each group has order 8 (i.e., each group contains 8 elements).

10. (1 pt each) TRUE / FALSE.

(a) **T** **F** If G is a group, $H \leq G$, and $a \in G$, then $Ha = H$ if and only if $a \in H$.

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(b) **T** **F** If H is a normal subgroup of the group G, then $ah = ha$ for each $a \in G$ and $h \in H$.

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(c) **T** **F** If H is a normal subgroup of G, then H is abelian.

(d) **T** **F** If G is abelian, then every subgroup of G is normal.

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(e) **T** **F** If G is a group having a prime number of elements, then G is cyclic.

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(f) **T** **F** For any two groups G and G' there is always a homomorphism $\varphi : G \to G'$.

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(g) **T** **F** $\mathbf{Z}_4 \times \mathbf{Z}_4$ is isomorphic to $\mathbf{Z}_{16}$.

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(h) **T** **F** There exist groups G and G' for which G is a subgroup of $(\mathbf{Z}, +)$, G' is a subgroup of $GL(2, \mathbf{R})$, and G is isomorphic to G'.

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(i) **T** **F** If H is a subgroup of the finite group G, and $a \in G$, then the left coset aH contains the same number of elements as the right coset Ha.

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(j) **T** **F** Suppose H is a normal subgroup of G. Then there is some group G', and a homomorphism $\varphi : G \to G'$, for which $H = Ker(\varphi)$.

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(k) **T** **F** There exists an onto homomorphism from $\mathbf{Z}$ to $\mathbf{Z}_4$.

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(l) **T** **F** The order of an element aH in the coset group G/H equals the number of elements in H.

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(m) **T** **F** If $H \leq G$ and $|H|$ is prime, then H is a cyclic subgroup of G.

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(n) **T** **F** If H is a subgroup of G, $|H| = 4$, and $a, b \in G$, then the set product of right cosets $Ha * Hb$ contains exactly 4 elements.