Math 414 Fall 2009 Abrams **Exam I** Name ______________________

Show all your work. No credit will be given for answers which are not accompanied by supporting computations. Make sure your notation is clear and precise. Use the back of the sheet if you need more space. Circle answers when appropriate. Throughout the exam, the letter e will always stand for the identity element of the group. Good luck !

1. (3 pt) Define precisely. A group $(G,*)$ is

2. (6 pt total) For each part of this question, either give a specific example or prove why no such example can exist. (The letter e always stands for the identity element of the group.)

a. A group G, and two distinct elements $a,b \in G$ for which $a*a=e$ and $b*b=e$.

b. A group G and an element $a \in G$ for which $a \neq e$ and $a^5 = a$.

c. A set S, and a binary operation $*$ on S, and three elements a,b,c in S, for which $a*b=a*c$ but $b \neq c$.

3. (2 pt) Show by giving a counterexample that the "proposed" binary operation $*$ on the set of rational numbers $\mathbf{Q}$ given by setting $\frac{a}{b} * \frac{c}{d} = \frac{a^2+c^2}{bd}$ is not well defined.

4. (1 pt) In the course of presenting a proof you write the sentence "We must show that the inverse is in the subset." Which of these two statements are you proving? (Circle one)

(i) The set G with binary operation $*$ is a group.

(ii) The subset H of the group G is a subgroup.

5. (4 pt total) a. Let G be any group. Define precisely what it means for the element a of G to be a *generator* of G.

b. Here are two abelian groups: $(\mathbf{Z}, +)$ (the integers) and V (the Klein 4 group). Of the two, one is cyclic, the other not. Circle the cyclic one. Then list all of the generators of the cyclic one.

6. (7 pt total) Let G be the group of 3×3 matrices with complex number entries having nonzero determinant, with matrix multiplication as the binary operation (i.e., $G = GL(3, \mathbf{C})$.) Let $K = \{A \in G \mid \det(A)$ is a nonzero real number$\}$.

a. Write down two specific elements of K. Explain for each why the element is in K.

b. Write down two specific elements of G which are not in K. Explain for each why the element is not in K.

c. Prove that K is a subgroup of G.

7. (6 pt total) Let n be any positive integer.

a. Define precisely what it means for an element τ of S_n to be *even*.

b. As usual, we let A_n denote the subset of S_n consisting of the even permutations. Prove that A_n is a subgroup of S_n. Make sure your notation is clear and precise.

8. (5 pt total) The group D_{10}.

a. $|D_{10}| =$

b. If we view D_{10} as a subgroup of S_{10} in the usual way, what is the permutation that corresponds to ”rotate through 144 degrees counterclockwise"? Give your answer in ’longhand’ permutation form. (You can label the vertices of the regular decagon counterclockwise.)

c. If we call the permutation in the previous part of this question σ, what is the order of σ in D_{10}?

d. Solve for i with $0 \le i \le 4 :\ \sigma^{18} = \sigma^{i}$.

e. Find σ^{-1}. Give your answer in ’longhand’ permutation form.

f. Find a POSITIVE integer t with the property that $\sigma^{-1} = \sigma^{t}$.

9. (4 pt total) Let $\alpha = (2,7,4,1)(4,8)(1,2,4)$ in S_8.

a. Write α in its 'longhand' form. $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ & & & & & & & \end{pmatrix}$

b. Write α as a product of disjoint cycles.

c. Write α as a product of transpositions.

d. **T** **F** $\alpha \in A_8$.

10. (3 pt total) In both of the parts of this question you are asked to find the order of an element in S_6. For one of them you can compute the order 'easily', using the result of a homework problem. For that one, do the 'easy' computation, and say WHY you can use the easy computation. For the other one, compute the order 'directly'.

a. Find the order of $(2,1,3)(2,3)$ in S_6.

b. Find the order of $(1,2,5)(3,4)$ in S_6.

11. (1 pt) For any finite group G and any element g of G, the smallest positive integer r for which $g^r = e$ is called ___________________ .

12. (1 pt) Give an example of two abelian groups K and L which each have the same number of elements, but which are not isomorphic to each other. (You need not justify your answer here, just give the two groups.)

13. (1 pt each) **True** / **False**.

a. **T** **F** If an infinite group G contains a subgroup H, then H must also be infinite.

b. **T** **F** If G is any group, $g \in G$, and $< g >$ denotes (as usual) the set $\{g^i \mid i$ is an integer$\}$, then $< g >$ is a subgroup of G.

c. **T** **F** If G is a group and $a \in G$ then there exists a positive integer r for which $a^r = e$.

d. **T** **F** It is possible to have a set S, and an equivalence relation ~ on S, for which there areonly finitely many distinct equivalence classes, and each equivalence class contains infinitely many elements of S.

e. **T** **F** It is possible to have a set S, and an equivalence relation ~ on S, for which there are infinitely many distinct equivalence classes, and each equivalence class contains infinitely many elements of S.