Fall 2009 Practice Exam for Exam 1

Math 414 [redacted] Abrams [redacted] Name SOLUTIONS

Show all your work. No credit will be given for answers which are not accompanied by supporting computations. Make sure your notation is clear and precise. Use the back of the sheet if you need more space. Circle answers when appropriate. Good luck !

1. (3 pt total) Define precisely. A group $(G,*)$ is

See p37

2. (5 pt total) Let G be any group, and let g be an element of G.
   a. Define precisely the subset $< g >$ of G.

$\{g^i \mid i \in \mathbb{Z}\}$

   b. Prove that $< g >$ is a subgroup of G. Use Subgroup Theorem

① Closure Pick $x, y \in \langle g \rangle$. Then $x = g^i$ $y = g^j$ for some $i, j \in \mathbb{Z}$ by def^n of $\langle g \rangle$ So $xy = g^i g^j = g^{i+j}$ $\in \langle g \rangle$ since $i+j \in \mathbb{Z}$.

② $e = g^0$ so $e \in \langle g \rangle$

③ If $x \in \langle g \rangle$ Then $x = g^i$ some $i \in \mathbb{Z}$. Then $x^{-1} = g^{-i}$ (since $g^i g^{-i} = g^0 = e$), and $g^{-i} \in \langle g \rangle$, so $x^{-1} \in \langle g \rangle$.

   c. For the particular case $G = Z_6$, find $< 5 >$.

$\{0, 5, 10, 15, 20, 25\} = \{0, 5, 4, 3, 2, 1\} = \mathbb{Z}_6$.

3. (5 pt total) Let $\alpha = (1,2,4)(2,8,7,6,1)(4,9)$ in S_9.

   a. Write α in its 'longhand' form. $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 8 & 3 & 9 & 5 & 2 & 6 & 7 & 1 \end{pmatrix}$

   b. Write α as a product of disjoint cycles.

$(1,4,9)(2,8,7,6)$

   c. Write α as a product of transpositions.

$(1,9)(1,4)(2,6)(2,7)(2,8)$

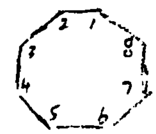
   d. T **F** $\alpha \in A_9$. Since α is The product of an odd number (here, 5) of transpositions

4. (4 pt total) The group D_8.

a. $|D_8| =$ $2 \cdot 8 = 16$

b. If we view D_8 as a subgroup of S_8 in the usual way, what is the permutation that corresponds to "rotate through 90 degrees counterclockwise"? Give your answer in 'longhand' permutation form. (You can label the vertices of the regular octagon counterclockwise.)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 6 & 7 & 8 & 1 & 2 \end{pmatrix}$$



c. If we call the permutation in the previous question σ, what is the order of σ in D_8?

Geometrically it's clear that $o(\sigma) = 4$. You can also crank it out

d. Find σ^{-1}. Give your answer in 'longhand' permutation form.

$\sigma^{-1} = \sigma^3$ since $o(\sigma) = 4$.
$\sigma^3 =$ rotate through $270°$ so $\quad \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 8 & 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$

e. Find a POSITIVE integer t with the property that $\sigma^{-1} = \sigma^t$.

$t = 3$ (previous problem)

5. (5 pt total) Let n be any positive integer.

a. Define precisely what it means for an element σ of S_n to be *even*.

σ can be written as the product of an even number of transpositions

b. As usual, we let A_n denote the subset of S_n consisting of the even permutations. Prove that A_n is a subgroup of S_n.

Use the "subgroup theorem"

1) Let $\sigma, \tau \in A_n$. Show $\sigma\tau \in A_n$.
By definition, σ can be written as the product of $2m$ transpositions (for some $m \in \mathbb{N}$).
Similarly, τ - - - - - - - - $2m'$ - - - - - (some $n \in \mathbb{N}$).
Then $\sigma\tau$ is the product of $2m + 2m' = 2(m+m')$ transpositions. Since $2(m+m')$ is even, we get $\sigma\tau \in A_n$

2) Since $e = (1,2)(1,2)$ is the product of 2 trans., and 2 is even, we have $e \in A_n$.

3) Suppose $\sigma \in A_n$. So $\sigma = (a_1, b_1)(a_2, b_2) \cdots (a_N, b_N)$ for some even number N. But by a general result on inverses,
$\sigma^{-1} = (a_N, b_N)^{-1} \cdots (a_2, b_2)^{-1}(a_1, b_1)^{-1} = (a_N, b_N) \cdots (a_2, b_2)(a_1, b_1)$
So σ^{-1} is the product of N transpositions. Since

6. Short Answer.

a. (1 pt) In the course of a proof you write the sentence "We must show that the operation is associative." Which of these two statements are you proving? (Circle one)

(i) The set G with binary operation $*$ is a group.

(ii) The subset H of the group G is a subgroup. (ii) comes for free for any subset of a group.

b. (2 pt) Give an example of an infinite abelian group which is not cyclic. Justify why the group is not cyclic. $(\mathbb{Q},+)$. Any cyclic subgroup must leave out some of the elements of $\mathbb{Q}$. For instance, if $x = r/s$ and $s = p_1^{k_1} \cdots p_t^{k_t}$ is the prime factorization of s, then for p a prime not equal to any p_i $1 \le i \le t$, $1/p \notin \langle r/s \rangle$.

c. (2 pt) Find the order of $(1,2,5)(2,3)$ in S_5.

$$(1,2,5)(2,3) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix}$$

$= (1,2,3,5)$. So the order is 4.

7. (7 pt total) Let G be the group of 2×2 matrices with real entries having nonzero determinant, with matrix multiplication as the operation (i.e., $G = GL(2,\mathbf{R})$.) Let $H = \{A \in G \mid A \text{ is lower triangular, and } \det(A) = \pm 1\}$.

(a) Write down three specific elements of H. There are ∞ many choices.

e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$, ...

(b) Prove that H is a subgroup of G. Use the Subgroup Theorem.

1) Let $A, B \in H$. The product of 2 lower Δ matrices is lower Δ, by a lin. alg. result. (or show directly!) also, $\det(AB) = \det(A)\det(B) = (\pm 1)(\pm 1) = \pm 1$.
So $AB \in H$.

2) $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in G. But e is lower Δ, and $\det(e) = 1$,
So $e \in H$.

3) Let $A \in H$. Since $\det(A) = \pm 1$ we have A^{-1} exists (since $\det(A) \neq 0$). $\det(A^{-1}) \overset{\text{Lin}}{\underset{\text{Alg}}{=}} 1/\det(A) = 1/(\pm 1) = \pm 1$.
Also, the inverse of a lower Δ matrix is lower Δ (Lin Alg). So $A^{-1} \in H$.

So $H \le G$.

8. (1 pt each) a. Let G be any group. Define precisely what it means for the element a of G to be a *generator* of G.

$G = \langle a \rangle$

b. List all the generators of the group $(\mathbf{Z},+)$.

$\{1, -1\}$

c. List all the generators of the group $(\mathbf{Z}_9,+)$.

$\{1,2,4,5,7,8\}$

d. Give an example of a group which contains no generators.

$(\mathbb{Q},+)$ $\quad V \cong \left\{ \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}-1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}-1&0\\0&-1\end{pmatrix} \right\}$ etc...

9. (5 pt total) (a) Let $(G,*)$ be a group. The identity element e of G is a solution to the equation $x^2 = x$, since $e^2 = e$. Prove that e is the ONLY solution to this equation in G.

Assume $f \in G$ has $f^2 = f$. Show $f = e$.
But $f^2 = f$. Perform f^{-1} on both: $f^{-1}f^2 = f^{-1}f$
So $f^1 = e$ So $f = e$. ∎

(b) Give an example of a group G, and an element $a \in G$, for which $a \neq e$, and a is a solution of the equation $x^3 = x$.

In $\mathbb{Z}_4$, $2+2+2 = 2$. $\underline{\text{or}}$ In $(\mathbb{C}^*,\cdot)$, $(-1)^3 = -1$.

10. (1 pt each) True / False.

a. **T** F An infinite group G could possibly contain a subgroup H which has only finitely many elements.

$\{e\}$ always works! $\underline{\text{or}}$ e.g. $\langle i \rangle$ in $(\mathbb{C}^*,\cdot)$

b. **T** F The group S_3 is not abelian.

c. **T** F D_4 is a subgroup of S_4.

d. **T** F For any finite group G and any element g of G, these three numbers are equal:
(i) The order of g (ii) $|\langle g \rangle|$ (iii) The smallest positive integer r for which $g^r = e$.

e. T **F** If G_1 and G_2 are groups, and $|G_1| = |G_2|$, then the groups G_1 and G_2 are isomorphic.

e.g. $|\mathbb{Z}_6| = |S_3| = 6$ but $\mathbb{Z}_6$, S_3 are not isomorphic

f. **T** F On the set of rational numbers $\mathbf{Q}$ we define a binary operation $*$ by setting $\frac{a}{b} * \frac{c}{d} = \frac{ac}{bd}$. Then $*$ is a well defined binary operation on $\mathbf{Q}$.

g. **T** F If G is a group which contains at least two elements, then G contains at least two unequal subgroups.

$\{e\}$ and G.