Quiz #1 - Solutions

On the front page, please PRINT your NAME and your student ID number.

This is NOT a multiple choice test, so **SHOW ALL YOUR WORK**. Correct answer with no work shown will receive **zero points**, while incorrect answers with correct work shown will receive partial credit! NO calculators are allowed in this test. Total points for this test is 25 if you correctly solve 2 out of 3 questions in each problem. If you solve more, you will get extra credit.

1. (**4 points: pick 2 out of 3 questions**) Give the equation of the lines described below. Write your answer in the $y = mx + b$ form.

   (a) The line which passes through the point $(-2,1)$ and has slope 3.

   Since we know the slope is $m = 3$, we can write the line as $y = 3x + b$ and then require that $(-2,1)$ satisfies the equation; this gives: $1 = 3\cdot(-2) + b$ so $b = 7$ and

   $$y = 3x + 7$$

   (b) The line which passes through the points $(3,1)$ and $(0,-1)$.

   The equation of a line through 2 points (x_1, y_1) and (x_2, y_2) (as we showed in class!) is $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$. In our case, we have

   $$\frac{y-1}{-1-1} = \frac{x-3}{0-3} \qquad \text{i.e.} \qquad \frac{y-1}{-2} = \frac{x-3}{-3}$$

   so $y - 1 = \frac{2}{3}(x-3)$ and finally

   $$y = \frac{2}{3}x - 1$$

   (c) The line that passes through $(1,2)$ and is parallel to the line $2y = 3x + 1$.

   The given line has slope $m = \frac{3}{2}$ (note the factor 2 multiplying y in the l.h.s!), and two parallel lines are characterized by having the same slope. Therefore, we can write the equation of the line we are looking for in the form $y = \frac{3}{2}x + b$. Then we replace the coordinates of the point $(1,2)$ to find b, getting: $2 = \frac{3}{2}\cdot 1 + b$ so $b = \frac{1}{2}$ and finally

   $$y = \frac{3}{2}x + \frac{1}{2}$$

2. (**4 points: pick 2 out of 3 questions**)

   (a) Simplify the following algebraic expression: $\left(\dfrac{x}{x+1}\right) / \left(\dfrac{x}{x+1} + 2\right)$

   Let us write

   $$\left(\frac{x}{x+1}\right) / \left(\frac{x}{x+1} + 2\right) = \frac{\dfrac{x}{x+1}}{\dfrac{x}{x+1} + 2}$$

The first thing to do is factor the denominator, which gives

$$\frac{\frac{x}{x+1}}{\frac{x}{x+1}+2} = \frac{\frac{x}{x+1}}{\frac{x+2(x+1)}{x+1}} = \frac{\frac{x}{x+1}}{\frac{3x+2}{x+1}}$$

Then we can invert the denominator and simplify, to find:

$$= \frac{x}{x+1}\cdot\frac{x+1}{3x+2} = \frac{x}{3x+2}$$

(b) If $f(r) = \pi r^{-2}$ for $r > 0$, find and simply $(f(r+2) - f(r))/2$.

Remember that $x^{-1} = 1/x$ and $x^{-n} = 1/x^n$. Therefore, $f(r) = \pi/r^2$ and we will write

$$(f(r+2) - f(r))/2 = \frac{1}{2}[f(r+2) - f(r)] = \frac{1}{2}\left[\frac{\pi}{(r+2)^2} - \frac{\pi}{r^2}\right] = \frac{\pi}{2}\left[\frac{1}{(r+2)^2} - \frac{1}{r^2}\right]$$
$$= \frac{\pi}{2}\frac{r^2 - (r+2)^2}{r^2(r+2)^2} = \frac{\pi}{2}\frac{-4r-4}{r^2(r+2)^2} = -2\pi\frac{r+1}{r^2(r+2)^2}$$

(c) If $f(r) = \pi r^2$, find and simply $\sqrt{f(t^3)}$.

The t is not a typo, I meant to stress that if I am given a function, then I am given "the rule", and all I have to do is apply that rule in correspondence of the argument that's required. Therefore, since the rule for f is: take the argument (r, or whatever you want to call it!), square it, then multiply it by π, to calculate $f(t^3)$, I take t^3, square it (which gives t^6), then multiply it by π, which finally gives πt^6. At this point, I need to take the $\sqrt{\ }$, so my final answer is:

$$\sqrt{f(t^3)} = \sqrt{\pi t^6} = \sqrt{\pi}\sqrt{t^6} = \sqrt{\pi}\,|t^3|$$

Note the absolute value!

3. (**4 points: pick 2 out of 3 questions**) Find all real solutions of the following equations.

(a) $\left(x^2 - 3\right) - \frac{3}{2}x(x+2) = 0$

Expanding all the products we have: $-\frac{1}{2}x^2 - 3 - 3x = 0$, which is equivalent to $x^2 + 6x + 6 = 0$. Then all I have to do is apply the formula for the zeros of a quadratic equation, to get $x_{1,2} = \frac{-6\pm\sqrt{36-24}}{2} = \frac{-6\pm\sqrt{12}}{2} = \frac{-6\pm2\sqrt{3}}{2}$. The TWO solutions of the equation are:

$$x = -3 + \sqrt{3} \qquad \text{and} \qquad x = -3 - \sqrt{3}$$

(b) $\sqrt{x^2+1} + 2 = 5$

First of all, I would look at the domain of the functions, to rule out values of x for which the equation doesn't even make sense. But in this case the argument of the square root is always ≥ 0, so the equation is defined for any $x \in \mathbb{R}$. Then I can write the equation as $\sqrt{x^2+1} = 3$ and since both terms are positive, I can square them both, getting $x^2 + 1 = 9$ so my two solutions are

$$x = \pm\sqrt{8} = \pm 2\sqrt{2}$$

(c) $\log_2 4 = 2x + 3$

The first thing to observe is that $\log_2 4 = \log_2 2^2 = 2\log_2 2 = 2$. Then the equation in simply a linear equation for x, $2x + 3 = 2$, whose solution is:

$$x = -1/2.$$

4. (**4 points: pick 2 out of 3 questions**) Find the exact values. Write your answer as a fraction, not as a decimal.

(a) $\sin\frac{4}{3}\pi$

I would observe that $\sin\frac{4}{3}\pi = \left(\pi + \frac{\pi}{3}\right)$ and therefore

$$\sin\frac{4}{3}\pi = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

(b) $\cos^2\dfrac{\pi}{6} + \tan^2\dfrac{\pi}{3}$

$$\cos^2\frac{\pi}{6} + \tan^2\frac{\pi}{3} = \left(\cos\frac{\pi}{6}\right)^2 + \frac{\left(\sin\frac{\pi}{3}\right)^2}{\left(\cos\frac{\pi}{3}\right)^2} = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{\left(\frac{1}{2}\right)^2} = \frac{3}{4} + \frac{3}{4}\frac{4}{1} = 3 + \frac{3}{4} = \frac{15}{4}$$

(c) $5e^{-\ln 3}$

$$5e^{-\ln 3} = 5\frac{1}{e^{\ln 3}} = 5 \cdot \frac{1}{3} = \frac{5}{3}$$

5. (**4 points: pick 2 out of 3 questions**) Solve the following inequalities for x.

(a) $\dfrac{2x-1}{5-x} \leq 0$

First of all, I will exclude $x = 5$ because for this value the equation does not even make sense. Then I consider separately the sign of numerator and denominator.

$$2x - 1 \geq 0 \quad \text{whenever} \quad x \geq 1/2$$
$$5 - x > 0 \quad \text{whenever} \quad x < 5$$

In order for the quotient to be negative, numerator and denominator have to have opposite signs, which happens if $x \leq 1/2$ or $x > 5$. In conclusion, the set of solutions is

$$]-\infty, 1/2] \cup]5, +\infty[$$

(b) $|x-4| \leq 3$

This inequality means $-3 \leq x-4 \leq 3$, which gives $1 \leq x \leq 7$ (just add 4 to all members of the inequality) and the set of solutions :

$$[1,7]$$

A different way to do it would be to remember the following: if $x-4 \geq 0$ (which is if $x \geq 4$), then $|x-4| = x-4$ and the inequality becomes: $x-4 \leq 3$, or $x \leq 7$ (so among the values of x that are larger than or equal to 4, I can only pick those that are smaller that 7). On the other hand, if $x-4 < 0$, (which is if $x < 4$), then $|x-4| = 4-x$ and the inequality becomes: $4-x < 3$, i.e. $x \geq 1$ (so among the values of x that are smaller than 4, I can only pick those that are larger than or equal to 1). Putting all the things together, I obviously get the same set of solutions, i.e. $[1,7]$.

(c) $2\sqrt[4]{x^{-3}} + 3 > -1$

As before, the first thing to do is make sure everything is defined in the expression. x can only take values that are strictly positive (because there is a $x^{-3} = 1/x^3$, which does not make sense for $x = 0$), and then there is a 4th root, which requires the argument to be ≥ 0. So the inequality is not even defined unless $x > 0$. At this point, I can rewrite it as

$$\sqrt[4]{x^{-3}} > -2$$

and then it is obvious that, no matter which value x takes (as long as the whole thing is defined), the $\sqrt[4]{}$ is going to be > 0, and hence > -2. So the set of solutions is:

$$]0,+\infty[$$

6. (**5 points: pick 2 out of 3 questions**) Find the domain of the following functions.

(a) $y = \sqrt[4]{x^2 + 8x + 7}$

Since I have a $\sqrt[4]{}$, its argument has to be ≥ 0. Then the domain of the function is defined by those x which satisfy

$$x^2 + 8x + 7 \geq 0$$

This is a quadratic inequality, whose graph is a parabola pointing upwards. I first find the values where the function vanishes (crosses the x-axis), through the quadratic formula:

$$x_{1,2} = \frac{-8 \pm \sqrt{64-28}}{2} = \frac{-8 \pm 6}{2} = -4 \pm 3 = \begin{cases} -7 \\ -1 \end{cases}$$

If you look at the graph of the parabola, it is clear that it is going to be ≥ 0 when $x \leq -7$ and when $x \geq -1$ (and negative in the middle). So the domain of the function is:

$$]-\infty,-7] \cup [-1,+\infty[$$

(b) $y = (1+\sin x)^{-1/2}$

Rewrite the function as $y = \dfrac{1}{\sqrt{1+\sin x}}$. Then we have to make sure $1 + \sin x > 0$ (so that I can take the square root first, and the reciprocal afterwards). But remember that $-1 \leq \sin x \leq 1$ for all x, so the only points that could make $1+\sin x > 0$, i.e. $\sin x < -1$, are those values of x for which $\sin x = -1$. In the interval $[0, 2\pi]$ this only happens when $x = \frac{3}{2}\pi$, so I have to exclude that point. And then, since $\sin$ is periodic of period 2π, I also have to exclude all other points $\frac{3}{2}\pi + 2\pi n$ for any integer n. The domain is:

$$\mathbb{R} \setminus \left\{ \frac{3}{2}\pi + 2\pi n : n \in \mathbb{R} \right\}$$

(c) $y = 1/\sqrt[3]{x^2 - 3x + 2}$

I have a $\sqrt[3]{\ }$, which is defined for any real value of its argument, but then I am taking the reciprocal, so I have to exclude the points where it vanishes, which are the points where its argument vanishes. So the domain is defined by all x such that $x^2 - 3x + 2 \neq 0$. Now

$$x^2 - 3x + 2 = 0$$

when $x_{1,2} = \dfrac{3 \pm \sqrt{9-8}}{2} = \dfrac{3 \pm 1}{2}$ so the domain is:

$$\mathbb{R} \setminus \{1, 2\}$$