Third Midterm - November 24, 2009

On the front page, please PRINT your NAME and your student ID number.

This is NOT a multiple choice test, so **SHOW ALL YOUR WORK**. Correct answer with no work shown will receive **zero points**, while incorrect answers with correct work shown will receive partial credit! Calculators are neither required nor allowed in this test.

1. (**60 points**) Use the guidelines to sketch the graph of the following function:

$$f(x) = 4\ln x + \frac{1}{8}x^2 - \frac{5}{2}x.$$

Skip A1, i.e. the study of the sign of the function, as well as x-intercepts. After you draw the graph, can you deduce some information about the solutions of the equation $f(x) = 0$ and the sign of $f(x)$?

A. domain: $\frac{1}{8}x^2 - \frac{5}{2}x$ is defined for all $x \in \mathbb{R}$; $4\ln x$ is not defined if $x \le 0$

hence $\boxed{\text{domain} = (0, +\infty)}$

A1. sign of $f(x)$: skip (it's a transcendental inequality)

B. $\boxed{\text{no y-intercept}}$ because $x=0$ not in domain; skip x-intercepts (as in A1)

C. 0 not in domain but function defined for any $x>0$ so we need to look at $\lim_{x\to 0^+} f(x)$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(4\ln x + \frac{1}{8}x^2 - \frac{5}{2}x\right) = 4\cdot \lim_{x\to 0^+} \ln x + \lim_{x\to 0^+}\left(\frac{1}{8}x^2 - \frac{5}{2}x\right) = 4\cdot(-\infty) + 0 = -\infty$

hence $x=0$ vertical asymptote from right: $\boxed{x=0 \downarrow}$

for horizontal asymptotes: the only limit that makes sense is $x \to +\infty$ ($-\infty$ does not make sense, as f is not defined for $x<0$)

$\lim_{x\to+\infty}\left(4\ln x + \frac{1}{8}x^2 - \frac{5}{2}x\right) = 4\cdot\lim_{x\to+\infty}\ln x + \lim_{x\to+\infty}\left(\frac{1}{8}x^2 - \frac{5}{2}x\right) = 4\cdot\lim_{x\to+\infty}\ln x + \lim_{x\to+\infty} \underset{\nearrow +\infty}{x^2}\underset{\searrow 0}{\left(\frac{1}{8} - \frac{5}{2x}\right)}$

$= +\infty + \infty = +\infty$ $\boxed{\text{no horizontal asymptote}}$

look for slant asymptotes: $\lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\frac{4\ln x + \frac{1}{8}x^2 - \frac{5}{2}x}{x} = \lim_{x\to+\infty} 4\frac{\ln x}{x} +$

$+ \frac{1}{8}\lim_{x\to+\infty} x - \frac{5}{2}$; for $\lim_{x\to+\infty} 4\frac{\ln x}{x}$ apply L'Hôpital's rule

$\lim_{x\to+\infty} 4\frac{\ln x}{x} = \lim_{x\to+\infty}\frac{4\cdot\frac{1}{x}}{1} = \lim_{x\to+\infty}\frac{4}{x} = 0$; hence $\lim_{x\to+\infty}\frac{f(x)}{x} = 0 + \infty - \frac{5}{2} = +\infty$

$\boxed{\text{no slant asymptote}}$

C. function $\boxed{\text{does not have any symmetry}}$, as it is not defined for $x<0$

E. $\boxed{f'(x) = \frac{4}{x} + \frac{2}{8}x - \frac{5}{2}} = \frac{4}{x} + \frac{x}{4} - \frac{5}{2} = \frac{16+x^2-10x}{4x}$

N.B. To proceed and study the sign, you always need to reduce to a common denominator and factor. Otherwise: how do you find the zeros???

F. $f'(x)>0 \iff \frac{16+x^2-10x}{4x} > 0 \iff \frac{(x-2)(x-8)}{4x} > 0$

note that $x>0$ in the domain, so DENOM $>0 \Rightarrow$ NUM >0 (for the quotient to be >0) $\Rightarrow (x-2)(x-8)>0$

f not defined!

0	2	8	
	- - -	+ + +	+ + + + +
	- - -	- - -	+ + + +
	+	-	+
	↗	↘	↗

$\boxed{f \text{ increasing in } (0,2) \text{ and } (8,+\infty)}$

$\boxed{f \text{ decreasing in } (2,8)}$

G. $x=2$ and $x=8$ are critical numbers (f' is zero at those points)

from F. I see $x=2$ is a local max, $x=8$ is a local min

$\boxed{f(2) = 4\ln 2 + \frac{1}{2} - 5 = \ln 2^4 - \frac{9}{2} = \ln 16 - 9/2 \quad \text{local max value}}$

$\boxed{f(8) = 4\cdot\ln 8 + 8 - 20 = \ln 8^4 - 12 \quad \text{local min value}}$

you can check that both $\ln 16 - 9/2 < 0$ and $\ln 8^4 - 12 < 0$

$\ln 16 - 9/2 < 0 \iff \ln 16 < 9/2 \iff e^{\ln 16} < e^{9/2} \iff 16 < e^{9/2} = \sqrt{e^9}$

(e^9 is a really big number $\sim 3^9$)

similarly, $\ln 8^4 - 12 < 0 \iff 8^4 < e^{12} \iff 2^{12} < e^{12}$ and again

this is true because $2<e$ so $2^{12}<e^{12}$

H. $\boxed{f''(x) = -\frac{4}{x^2} + \frac{1}{4}} = \frac{-16+x^2}{4x^2} = \frac{(x-4)(x+4)}{4x^2}$

$f''(x)>0 \iff \frac{(x-4)(x+4)}{4x^2} > 0$ DENOM >0 so NUM has to be >0

$\iff (x-4)(x+4)>0$ since $x>0$, $x+4>0$, hence $x-4>0$

so $f''(x)>0 \iff x-4>0$ (for $x>0$)

equivalently:

-4	0	4	
$x-4>0$	- - -	- - -	+ + + +
$x+4>0$	- -	+ + +	+ + + + +
+		- ∩	+ ∪

$\boxed{f \text{ is concave upward in } (4,+\infty)}$

$\boxed{f \text{ is concave downward in } (0,4)}$

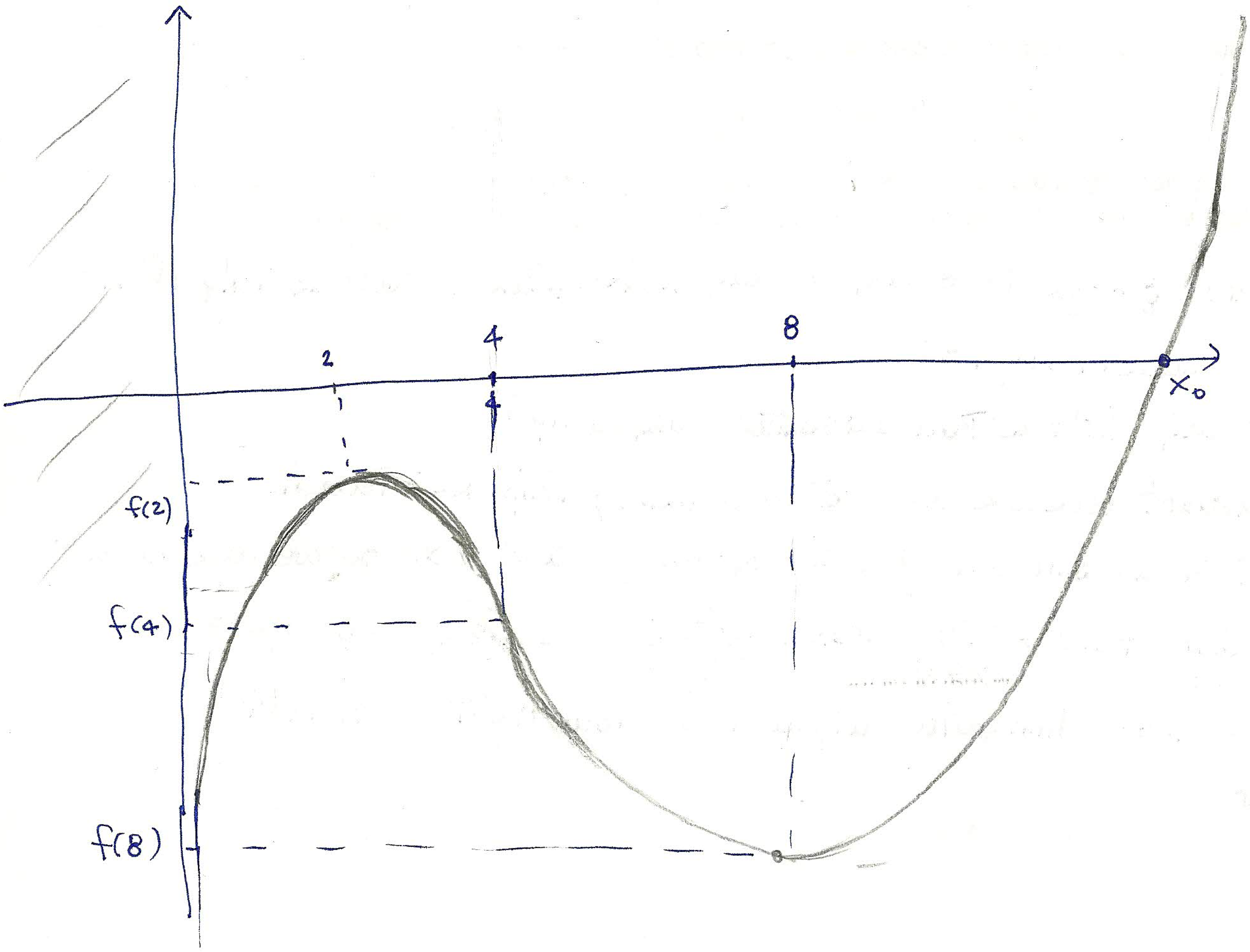
$\boxed{x = 4 \text{ is an inflection point}}$

$\boxed{f(4) = 4 \ln 4 + 2 - 10 = \ln 4^4 - 8 = \ln 2^8 - 8}$

again you can check $\boxed{f(4) < 0}$: $\ln 2^8 - 8 < 0 \iff e^{\ln 2^8} < e^8 \iff$

$2^8 < e^8$ which is True because $2 < e$

Note the function does not have any global max or min because in D we showed $\lim_{x \to 0^+} f(x) = -\infty$, $\lim_{x \to +\infty} f(x) = +\infty$



from the graph you see that the function crosses the x-axis only once, for some $x_0 > 8$. Hence the equation $f(x) = 0$ has only one solution, x_0, which we could find by Newton's method picking a starting point > 8.

The function $f(x) < 0$ for $x < x_0$, and $f(x) > 0$ for $x > x_0$.